Exercises to the course on
"Large deviations, moments problems and sum rules"

1 Moments and spectral measures

Exercise 1: Show that the n-th moment space

$$\mathcal{M}_n([0,1]) = \left\{(m_1(\mu), \dots, m_n(\mu)) : m_k(\mu) = \int x^k d\mu(x), \mu \text{ p.m. on } [0,1]\right\}$$

is the convex hull of

$$C_n = \{(x, x^2, \dots, x^n) : x \in [0,1]\}.$$

Exercise 2: Let μ be the spectral measure of

$$J = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & \ddots & \ddots \\ & & \ddots & \end{pmatrix},$$

that is, with moments $m_k(\mu) = \langle e_1, J^k e_1 \rangle$.

1. Show that for polynomials p, q, we have $\langle p, q \rangle_{L^2(\mu)} = \langle p(J)e_1, q(J)e_1 \rangle$.
2. Show that if $p_0, p_1, p_2, \dots$ are the polynomials orthonormal w.r.t μ, then (with $a_0 = 0$)

$$xp_k(x) = a_{k+1}p_{k+1}(x) + b_{k+1}p_k(x) + a_k p_{k-1}(x).$$

3. Show that if $b_k = 0$ and $a_k = 1$ for all k, then $m_{2k-1}(\mu) = 0$ and $m_{2k}(\mu) = \binom{2k}{k} - \binom{2k}{k-1}$.

2 Large deviations

Exercise 3: Suppose that $(X_n)_n$ satisfies an LDP with speed a_n and rate function $\mathcal{I}$ and $b_n/a_n \to 0$. Prove that then $(X_n)_n$ satisfies also an LDP with speed b_n and determine the rate function.

Exercise 4: Show that if $(X_n)_n$ satisfies the LDP with speed a_n and rate function $\mathcal{I}$, then

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{a_n} \log \mathbb{P}(X_n \in B_\varepsilon(x)) = -\mathcal{I}(x)$$

Exercise 5: Suppose X_n has a Lebesgue density on a bounded open set $U \subset \mathbb{R}^d$ given by

$$C_n \exp\{-a_n G(x)\},$$

where G is continuous with compact level sets and $\inf_x G(x) = 0$. Show that (X_n) satisfies the LDP with speed a_n and rate function G.

Exercise 6: Show that the Kullback-Leibler divergence can be written as

$$\mathcal{K}(\mu|\nu) = -\int \log(w)\, d\mu$$

when μ, ν are probability measures on $\mathbb{R}$ such that the Lebesgue decomposition of ν with respect to μ is

$$d\nu = w d\mu + d\mu_s.$$

3 Random matrices

Exercise 7: Let X_n by an $n \times n$ random matrix of the Gaussian unitary ensemble with spectral measure μ_n. Derive large deviation principles for the first two moments $m_1(\mu_n)$ and $m_2(\mu_n)$.

Exercise 8: Show that the Gaussian unitary ensemble is invariant under the conjugations $X \mapsto UXU^*$ for U a unitary matrix.
Hint: The conjugation is a linear mapping in the real independent entries of X. Also $\operatorname{tr}(X^2) = \operatorname{tr}((UXU^*)^2)$.

Exercise 9: Let $Z_1, \ldots, Z_n$ be independent, Exp(1) distributed random variables. Determine the distribution of

$$\left(\frac{Z_1}{Z_1 + \cdots + Z_n}, \ldots, \frac{Z_n}{Z_1 + \cdots + Z_n}\right)$$

Exercise 10: Show that for the Gaussian unitary ensemble, the weighted spectral measure $\mu_n = \sum_k w_k \delta_{\lambda_k}$ has the same weak limit (in probability) as the empirical eigenvalue measures $\hat{\mu}_n = \frac{1}{n} \sum_k \delta_{\lambda_k}$.