Lecture 1
Malliavin-Skorohod calculus in Finance and Insurance

Josep Vives *(josep.vives@ub.edu)*

Departament de Matemàtica Econòmica, Financera i Actuarial

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Mallavin-Skorohod calculus

Malliavin-Skorohod calculus is a stochastic calculus, an alternative and an extension of Itô calculus, that allows to manage non-adapted stochastic processes.

It is based on two dual operators, a *derivative* D and an *integral* δ.

Two formulas related with these operators have been proven to be extremely useful in applications:

- The Clark-Haussmann-Ocone formula, that extends the predictable representation theorem of functionals of a martingale, allowing to explicitly identify the predictable kernel. It is a type of fundamental calculus theorem.
- An (infinite dimensional) integration by parts formula.

Basic ingredients

In the context of Gaussian processes:

- The Malliavin derivative D was introduced in **P. Malliavin** (1978): *Stochastic calculus of variations and hypoelliptic operators*. Proceedings of the International Symposium on Stochastic Differential Equations, Kyoto University, 1976: 195-263. Wiley.
- The Skorohod integral δ was introduced in **A. V. Skorohod** (1975): *On generalization of the stochastic integral*. Theory of Probability and Applications 20: 219-233.
- The duality between the two previous operators was established in **B. Gaveau and P. Trauber** (1982): *L'intégrale stochastique comme operateur de divergence dans l'espace fonctionnel*. Journal of Functional Analysis 46: 230-238.

Essential History

Malliavin-Skorohod calculus was developed

- For functionals of a Gaussian process during the eighties, see for example Nualart-Zakai (1986) and Nualart-Pardoux (1988).
- For functionals of a Poisson process during the nineties, see for example Nualart-Vives (1990), Privault (1993), Nualart-Vives (1995), Picard (1996) and others.
- For functionals of a Lévy process or, more generally, of an additive processes, during the first decade of this century. See for example León-Solé-Utzet-Vives (2002), Benth-Di Nunno-Løkka-Øksendal-Proske (2003), Solé-Utzet-Vives (2007a, 2007b), and many others.

Applications to Finance

The purpose of this introductory Lecture 1 is to show how Malliavin-Skorohod calculus is a useful tool to treat some important problems in Quantitative Finance.

During the last thirty years, Malliavin-Skorohod calculus has been applied to different topics in Finance and Insurance. Between them, we emphasize

- Pricing and hedging financial derivatives.
- Computation of Greeks.
- Analysis of the volatility surface.
- Pricing cumulative loss derivatives in Insurance.

In this first introductory lecture I want to recall some of the more established applications.

History Notes: first application

Applications of Malliavin-Skorohod calculus to Finance have been developed during the last thirty or thirty-five years.

Probably the first one was **I. Karatzas and D. Ocone** (1991): *A generalized Clark representation formula with applications to optimal portfolios*. Stochastics and Stochastics Reports 34: 187-220, where an elegant solution of the problem of pricing and hedging financial derivatives in complete markets was found using the currently so called Clark-Haussman-Ocone formula.

These ideas are well explained in **B. Øksendal** (1996): *An introduction to Malliavin calculus with applications to economics. Working paper of the Norwegian School of Economics and Business Administration 3/96*.

Second application

A second key application appeared in 1999 with the celebrated paper

E. Fournié, J. M. Lasry, J. Lebuchoux, P. L Lions and N. Touzi (1999): *Applications of Malliavin calculus to Monte Carlo methods in Finance*. Finance and Stochastics 3 (4): 391-412,

where the integration by parts formula was applied successfully to improve the efficiency in computing Greeks, reducing dramatically the numerical difficulties of this type of computations.

Third application

A third interesting application was

E. Alòs (2006): *A generalization of the Hull and White formula with applications to option pricing approximation*. Finance and Stochastics 10 (3): 353-365,

where Malliavin-Skorohod calculus, and specially an anticipating Itô formula, was applied to obtain an expansion of the pricing formula under stochastic volatility diffusion models, that allows to distinguish clearly the effect of correlation in prices.

This formula, an extension of the classical Hull and White formula, allows to obtain interesting results related with the shape of the implied volatility surface, see Alòs-León-Vives (2007) and Alòs-León-Pontier-Vives (2008).

Fourth application

A fourth application is the use of the integration by parts formula for pricing cumulative loss derivatives. Recall that the cumulative loss process is a process

$$L_t := \sum_{i=1}^{N_t} Y_i, \; t \geq 0, \; L_0 = 0, a.s.$$

where N is a counting process that describes the arrival of claims and Y_i, for $i \geq 1$, are positive random variables that describe the size or the amount of claims.

The integration by parts formula based on Malliavin-Skorohod calculus techniques for additive processes helps us to compute quantities like $\mathbb{E}(H(L_T))$, for different suitable functions H and for different models for the cumulative loss process L.

Prices and returns I

- In a free market, prices evolve under the law of supply and demand, in a random way. So, they can be seen as observed realizations of a stochastic process. Therefore, the Theory of Stochastic Processes is an important tool in Quantitative Finance.
- To fix ideas, assume we observe prices $s_0, s_1, \ldots, s_n$ of a security at a certain fixed time instants $t_0, t_1, \ldots, t_n$ in a finite time period $[0, T]$. Assume that $t_0 = 0$, $t_n = T$ and $\Delta := t_i - t_{i-1}$ is constant, that is, times $t_1, \ldots, t_n$ are equally spaced. Say, for example, that $s_0, s_1, \ldots, s_n$ are daily closing prices.
- To model mathematically this time series of prices we can introduce a continuous time stochastic process $S := \{S_t, t \geq 0\}$ in such a way that every S_t be a random variable that represents the price at t and so, $(s_0, \ldots, s_n)$ is a realization of $(S_{t_0}, \ldots, S_{t_n})$.

Prices and returns II

Daily variation of a price is of course a quantity relative to the order of magnitude of the price. Therefore, it is obvious that the relevant quantities when we analyze price variations cannot be the absolute variations but the relative ones, the so called returns

$$r_i := \frac{s_i - s_{i-1}}{s_{i-1}} = \frac{s_i}{s_{i-1}} - 1.$$

Note that we can also write $s_i = s_{i-1}(1 + r_i)$ and $\log s_i - \log s_{i-1} = \log(1 + r_i)$.

Prices and returns III

Henceforth, process X defined as $X_t := \log S_t$, represents the so called log-price process and its observed values at $t_0, t_1, \ldots, t_n$ are $x_i := \log s_i$. Moreover we can consider the log-returns $y_i := \log s_i - \log s_{i-1} = \log(1 + r_i)$ that are the increments of process X.

Recall that as a consequence of Taylor formula we have

$$\log(1 + r) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{r^k}{k} = r - \frac{r^2}{2} + \cdots ,$$

and so, for values r_i not far from 0 ($|r_i| < 0.2$), we have $y_i \approx r_i$.

Note also that if $r_n(k)$ represents the return after k days, or k periods of time, we have

$$(1 + r_n(k)) = \frac{s_n}{s_{n-k}} = \frac{s_n}{s_{n-1}} \cdots \frac{s_{n-k+1}}{s_{n-k}} = (1 + r_n) \cdots (1 + r_{n-k+1})$$

and taking logarithms $y_n(k) := \log(1 + r_n(k)) = y_n + \cdots + y_{n-k+1}$.

Prices and returns IV

Thus, the general problem of modeling prices can be reduced to built probabilistic models to describe log-returns y_i that are nothing more than increments, or first differences, of the log price process X.

There is another natural reason to model log prices better than prices itself. In free markets we have interest rates, that is, for a given currency with fixed interest rate r, a unit at time 0 becomes e^{rt} units at time t. So, prices written in a concrete currency have an underlying exponential trend as currencies do. To model log prices allows to manage the interest rate as a linear trend.

In this course we assume for simplicity a fixed interest rate $r \geq 0$.

Prices and returns V

As a summary, in Finance, we are interested in the following three mathematical objects:

1. The price process $S := \{S_t, t \geq 0\}$.
2. The log-price process $X := \{X_t, t \geq 0\}$.
3. The increments of the log-price process $Y := \{Y_n, n \geq 1\}$ where

$$Y_n := X_n - X_{n-1} = \log S_n - \log S_{n-1},$$

Therefore, the question is what can we say about random variables Y_n or about processes X and S. How to describe them in a probabilistic way?

The Osborne-Samuelson model I

Since the sixties, the standard model, or the benchmark, for prices, is the so called Osborne-Samuelson model, see Osborne (1959) and Samuelson (1965). This model assumes that price process S evolves as a Geometric Brownian motion, that is,

$$S_t = s_0 e^{\alpha t} e^{\sigma W_t},\ t \geq 0,$$

where s_0 is the current price, α is a real parameter, σ is a positive parameter and W is a standard Brownian motion. Note that this is equivalent to assume

$$X_t = x_0 + \alpha t + \sigma W_t$$

where $x_0 := \log s_0$.

Taking equally spaced observations we have the increments

$$Y_n := \alpha + \sigma(W_n - W_{n-1}),\ n \geq 1.$$

The Osborne-Samuelson model II

Recall that a standard Brownian motion W is a centered process, null at the origin, with independent and stationary increments, such that the increments have a centered normal law with variance the increment of time. Therefore, $W_n - W_{n-1}$ are independent and identically distributed random variables with standard normal law.

So, according the Osborne-Samuelson model, a series of daily log-returns $y_i := x_i - x_{i-1}$ has to be a sample of a normal random variable with mean α and standard deviation σ. Is this model supported by empirical observations?

As we are going to see, the answer, on the basis of the so called stylized facts, is clearly negative. The only point in favor of this model is the so called aggregated normality phenomenon, which shows that more the increments are large (monthly, yearly), more the histogram of observations $y_1, \ldots, y_n$ is close to the normal density.

Characterization of the Brownian motion I

The mathematical support of Osborne-Samuelson model is the following theorem.

Theorem: Let $Z := \{Z_t, t \geq 0\}$ be a stochastic process such that $Z_0 = z_0 \in \mathbb{R}$ a.s. Assume the following hypotheses:

- H1) Z has independent increments, that is $Z_{t+h} - Z_t$ is independent of the $\sigma-$algebra generated by $\{Z_s, s \leq t\}$, for any $t, h \geq 0$.
- H2) Z has stationary increments, that is, the law of increments $Z_{t+h} - Z_t$ and Z_h is the same, for any $t, h \geq 0$.
- H3) Z has a.s. continuous trajectories.

Then, there exists $\alpha \in \mathbb{R}$ and $\sigma \geq 0$ such that $Z_t = z_0 + \alpha t + \sigma W_t$ where W is a standard Brownian motion. ($W_t - W_s \sim N(0, t-s)$).

Characterization of the Brownian motion II

The proof of this result is immediate as a corollary of the Lévy-Itô decomposition that we will see in Lecture 3. A direct proof can be found in Bouleau (2000). See also Lamberton-Lapeyre (2007).

Therefore, Osborne-Samuelson model is a good model if hypotheses (H1)-(H3) are reasonable, and it is not a good model if at least, one of the three hypotheses is far to be true. Are hypotheses (H1)-(H3) reasonable?

The stylized facts

- Log-return series show a similar behaviour for many different type of financial products. The typical phenomena observed in these series are called in the literature, the stylized facts.
- The easiest way to check Osborne-Samuelson hypotheses is to test the normality of a sample of increments of a log-price process observed in equally spaced time instants. Unfortunately, the hypothesis of normality is typically rejected. Why? Which of the hypotheses (H1)-(H3) is not reasonable? There are arguments to think that in fact, any of them is true.
- In general, the log price process, and the corresponding sample of daily log returns, show frequently the stylized facts listed below. See for example Rydberg (1997), Rydberg (2000), Schoutens (2003) and Cont-Tankov (2004).

Intermittency

- In series of log-returns we observe a high degree of variability, with periods of large positive values followed by large negative values. This fact weaken the hypothesis of continuity of trajectories.
- Jumps seem to be able to explain better the sudden strong changes of the log price process in short periods of time that we observe from time to time.
- In short, (H3) could be false; in any case it is not a hypothesis with a strong support.

Volatility clustering

- If we measure the volatility, a financial way to name the standard deviation, in different disjoint periods of a sample of log prices, sequentially, the values turn to be positively correlated.
- This is the so called volatility clustering effect, high volatility periods appear to be clustered in time.
- This effect suggest that volatility is not constant and so hypothesis (H2), that says that log returns are stationary, seems not to be strongly supported.

Uncorrelation and Taylor effect

- Certainly, no significance is typically found in correlation tests between log-returns, but on the contrary, absolute values of log returns, or powers of them, appears to be correlated. This is the so called Taylor effect, and this is a major drawback.
- This implies that (H1), that is, independence of log-returns, cannot be assumed, and neither normality, because uncorrelation in a normal framework implies independence.
- Moreover, this makes histograms of log-returns to be not useful. A histogram has complete sense only if it is based on a sample, that is, on a series of independent and identically distributed data; if not, the sense is unclear.

Long range dependence / Leverage effect / Skewness

- Sometimes, but not always, auto-correlation of absolute log-returns decay slowly enough as a function of time lag, a fact that suggests long range dependence.
- Volatility is usually negatively correlated with returns. This can be easily explained by the idea that downward movements of prices imply increases of the volatility.
- In equities and indexes large downwards are more frequent that large upwards. This is easily explained by the idea that panic is stronger than euphoria, because panic is less corrected by skepticism. This generates a skeweness of log price values. This not happens on currencies because a large downward of one currency is a large upward of the other.

Summary

Later we will see more drawbacks related to pricing and hedging financial derivatives, specially related with volatility.

But in any case it seems clear that it is necessary to go beyond the Osborne-Samuelson model to describe correctly log-price processes and to develop methods of pricing and hedging derivatives based on these more general models as we see in next lectures.

Beyond Osborne-Samuelson model

In order to go beyond Osborne-Samuelson model we can follow different strategies.

- To change hypothesis (H3), the continuity of trajectories, by the weak hypothesis of right continuous with left limits (càdlàg) trajectories. In this case, by definition, process Z becomes a Lévy process.
- To remove moreover (H2), the stationary character of increments, and add the technical hypothesis of continuity in probability, (nothing to do with continuity of trajectories). In this case we obtain by definition that Z is an additive process.
- Certainly, independence of increments is neither true. This is the major problem of Empirical Finance. But we can consider additive processes as building blocks in the construction of more sophisticated approaches like stochastic volatility jump diffusion models, that have no independent increments.

Financial Derivatives I

Black and Scholes solution of the problem of pricing and hedging financial derivatives on the basis of the Osborne-Samuelson price model, done in Black-Scholes (1973), triggered the contemporary theory of Quantitative Finance based on the principle of No Arbitrage (NA).

The main financial application of the mathematics developed in this course are related with the problem of pricing and hedging financial derivatives.

Financial Derivatives II

The basic examples of a financial derivative are European plain vanilla call and put options.

An European plain vanilla call option gives to the owner the right, but not the obligation, to buy a certain amount of a good quoted in a market, at a certain price $K \geq 0$ in a certain future date $T \geq 0$.

At date T, if the price $S_T \leq K$, the option is not exercised because to buy at price K is nonsense if you can buy the good for the minor price S_T in the spot market. On the contrary, if $S_T > K$ the owner will exercise the option and buy the good for K.

Of course, the owner can sell the good immediately in the spot market and obtain a benefit of $S_T - K$.

Financial Derivatives III

Note that for the owner of the option, it is equivalent to obtain the good paying K than to obtain $S_T - K$ euros in cash, exercising the option, and jointly with their K euros, to buy the good in the spot market.

Accordingly with previous comments we will write the payoff of an European plain vanilla Call as $(S_T - K)^+$. Similarly, the payoff of a European plain vanilla Put will be $(K - S_T)^+$.

Financial Derivatives IV

So, essentially, an option, or more generally, an European type financial derivative, is anything more than the right to receive a certain amount of money at a certain future date. An amount of money that is different for any possible evolution of the underlying price in the market. In this sense, it is an unpredictable or random gain.

Note that in fact we can enlarge the abstract point of view enough to include for example futures and forwards if we understand that to gain can mean also to loose (negative gain) for some future scenarios.

Recall, on other hand, that European type means that the derivative can be exercised only at the maturity date T. On the contrary, the so-called American-type options can be exercised at any time during its lifetime. All the derivatives considered in this course will be of European type.

Financial Derivatives V

Let $S := \{S_t, t \in [0, T]\}$ be the stochastic process that describes the evolution of the underlying price during the lifetime of the option. Let $\mathbb{F} := \{\mathcal{F}_t, t \in [0, T]\}$ the completed natural filtration associated to S. Recall that a filtration is a mathematical object that describes the information available at any time.

Mathematically, an European type financial derivative with expiry date T, written over S, is a random variable H_T, $\mathcal{F}_T-$measurable. Recall that in general, to say that a random variable H is $\mathcal{F}_T-$ measurable means in particular that the value of H is known at T.

Spot and path dependent derivatives

From the mathematical point of view, there are two major types of European type financial derivatives:

- Derivatives that depend only on S_T, that is, derivatives with payoff $f(S_T)$, for some measurable function f,
- The so called path-dependent derivatives, that depend on the whole previous trajectory of S, or on some part of it.

Examples of financial derivatives

- Examples of spot dependent derivatives are plain vanilla call and put options, with payoffs $H_T := (S_T - K)^+$ and $H_T := (K - S_T)^+$ respectively, or digital options with payoffs like $H_T := \mathbf{1}_{[K,\infty)}(S_T)$, or forward contracts, that can be seen as a derivative with payoff $H_T := S_T - K$.
- Examples of path-dependent options are look-back options with a payoff like

$$H_T := (\max_{t\in[0,T]} S_t - K)^+,$$

barrier options with a payoff like

$$H_T := (S_T - K)^+ \mathbf{1}_{\{\max_{t\in[0,T]} S_t \leq M\}}$$

where M is a constant upper barrier, or Asian options with a payoff like

$$H_T := (\frac{1}{T}\int_0^T S_t dt - K)^+.$$

The problem of pricing and hedging

- The main question related with financial derivatives is the problem of pricing and hedging them.
- If I buy a derivative I buy the right to obtain a profile of gains, depending on possible evolutions of the market, at a certain date T. What have I to pay for this right? What is the prime of this contingent claim? This is the problem of pricing.
- But, for the writer of the derivative, the question is, given the prime, what I have to do with it, in order to be able to cover or hedge the profile of payments I will be obliged to satisfy at T? This is the problem of hedging.
- Malliavin-Skorohod calculus is a calculus for functionals of some stochastic processes. Recall that a financial derivative H_T is a functional of the stochastic process S.

Gaussian Malliavin calculus operators

- Assume we have a standard Brownian motion $W := \{W_t, t \geq 0\}$ defined on a certain filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F}$ is assumed to be the completed natural filtration of W.
- The Malliavin-Skorohod calculus introduces two operators, the so-called Malliavin derivative D and the so-called Skorohod integral δ.
- Operator δ is the adjoint of operator D and can be interpreted as an extension of the Itô stochastic integral to integrands not necessarily adapted to the underlying filtration.
- Operators D and δ act on functionals $F \in L^2(\Omega)$ and processes $u \in L^2(\Omega \times [0, \infty))$, respectively. These hypothesis can be weakened in some cases.

Duality

Given u and F in suitable domains, operators D and δ satisfy the following duality relationship:

$$\mathbb{E}(\langle u, DF\rangle) = \mathbb{E}(\delta(u)F). \tag{1}$$

where

$$\langle u, DF\rangle := \int_0^\infty u_s D_s F ds.$$

CHO formula

Applications of the Malliavin-Skorohod calculus increased by means of Clark-Haussmann-Ocone (CHO) formula, proved in Ocone (1984), and based on previous references Clark (1970) and Haussmann (1979).

It shows that Malliavin-Skorohod calculus is able to identify the kernel of the martingale representation theorem. Concretely, if $F \in L^2(\Omega)$ is a $\mathcal{F}_T-$measurable functional of the Brownian motion W, and it is in the domain of operator D, we have

$$F = \mathbb{E}(F) + \int_0^T E[D_t F | \mathcal{F}_t] dW(t), \tag{2}$$

where the integral is of Itô type and $E[D_t F | \mathcal{F}_t]$ is a stochastic process that for every t (almost everywhere) denotes the conditional expectation of the random variable $D_t F$ with respect the $\sigma-$algebra $\mathcal{F}_t$.

As we will see, formula (2) is the basic tool for pricing and hedging financial derivatives.

The integration by parts formula I

Assume the price process S depends on a parameter θ. Choose $F = g(S_T^\theta)$ where g is a derivable function and consider u a certain process.

It can be proved, in the Gaussian setting (but not necessarily in other cases), the chain rule

$$DF = g'(S_T^\theta) DS_T^\theta.$$

Then,

$$\langle Dg(S_T^\theta), u \rangle = g'(S_T^\theta) \langle DS_T^\theta, u \rangle.$$

Therefore, we have,

$$\partial_\theta \mathbb{E}[g(S_T^\theta)] = \mathbb{E}[g'(S_T^\theta) \partial_\theta S_T^\theta] = \mathbb{E}[\frac{\langle Dg(S_T^\theta), u \rangle}{\langle DS_T^\theta, u \rangle} \partial_\theta S_T^\theta] = \mathbb{E}[\langle Dg(S_T^\theta), \frac{u \cdot \partial_\theta S_T^\theta}{\langle DS_T^\theta, u \rangle} \rangle]$$

The integration by parts formula II

Then, from (1), we obtain

$$\partial_\theta \mathbb{E}(g(S_T^\theta) = \mathbb{E}[g(S_T^\theta) \cdot \delta(\frac{u \cdot \partial_\theta S_T^\theta}{\langle DS_T^\theta, u\rangle})]. \tag{3}$$

If we denote $\pi_T := \delta(\frac{u \cdot \partial_\theta S_T^\theta}{\langle DS_T^\theta, u\rangle})$, the formula takes the form

$$\partial_\theta \mathbb{E}(g(S_T^\theta)) = \mathbb{E}[g(S_T^\theta) \cdot \pi_T] \tag{4}$$

Formula (4) has proved to be very useful in numerical computation of Greeks, as it is shown in Fournié-Lasry-Lebuchoux-Lions-Touzi (1999).

Note that it converts the problem to compute a derivative to the problem to compute a product by a certain weight.

Pricing and hedging financial derivatives I

- Pricing is the problem to determine what is the fair price V_0 for the right to obtain $H_T(\omega)$ monetary units at the date T, where different $\omega \in \Omega$ denote different possible evolutions of the underlying price S.
- Hedging is the problem to determine what the writer or seller of the option has to do with the prime V_0 in order to replicate at T the random payoff H_T for every possible scenario ω.

Pricing and hedging financial derivatives II

- Pricing and hedging financial derivatives is an important goal in Finance. It was elegantly achieved in Black-Scholes (1973) assuming that prices follow the Osborne-Samuelson model and the no arbitrage principle.
- They reduced the problem to solve a partial differential equation that was essentially the heat equation.
- Some years later, in Harrison-Pliska (1981), the authors considered the problem again using a probabilistic point of view based on the theory of martingales and Markov processes.
- The connection between the two points of view is the famous Feynman-Kac formula.

Pricing and hedging financial derivatives III

- Malliavin-Skorohod calculus provide a tool to analyze the question of pricing and hedging financial derivatives since a third point of view that avoids the hypothesis of markovianity.
- Under the Osborne-Samuelson model, we present here the skeleton of this point of view.
- The original reference is Karatzas-Ocone (1991). The same development is presented in Øksendal (1996) and in Nualart (2006).
- The course is devoted to extend this approach beyond the Osborne-Samuelson model.

Pricing and hedging financial derivatives IV

Assume S follows the Osborne-Samuelson model. In terms of a solution of an stochastic differential equation it can be written as

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

where $\mu = \alpha + \frac{\sigma^2}{2}$. This is a straightforward consequence of Itô formula.

The classical solution of pricing and hedging a financial derivative with payoff G consists in create a self-financing dynamic portfolio such that at time t it has a_t units of the underlying asset and b_t units in a bank account in such a way that the value V_t at t is

$$V_t = a_t S_t + b_t e^{rt}$$

where V_0 is a suitable constant and $V_T = G$.

Pricing and hedging financial derivatives V

The self-financing condition implies that no money is added to or subtracted from the portfolio once it is established, and therefore, its changes depend only on the random changes of process S. Mathematically,

$$dV_t = a_t dS_t + b_t e^{rt} r dt.$$

Consider the discounted quantities $\tilde{S}_t := e^{-rt} S_t$ and $\tilde{V}_t := e^{-rt} V_t$. Applying another time the Itô formula we obtain

$$d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t$$

and

$$d\tilde{V}_t = a_t d\tilde{S}_t. \tag{5}$$

Pricing and hedging financial derivatives VI

This last equation can be written in integral form as

$$\tilde{V}_t = V_0 + \int_0^t a_u d\tilde{S}_u$$

where the integral has to be understood as an Itô integral with respect the semi-martingale $\tilde{S}$.

Therefore, the problem of pricing and hedging G reduces to obtain a positive constant V_0, that will be the price, and a predictable process a, that will be the hedging strategy, such that

$$\tilde{G} = V_0 + \int_0^T a_u d\tilde{S}_u \tag{6}$$

where $\tilde{G} := e^{-rT}G$.

Pricing and hedging financial derivatives VII

How can we solve this problem? And moreover, can the solution be extended to models more general than the Osborne-Samuelson one?

The solution under the Osborne-Samuelson model is based on two major results of the Itô stochastic calculus for Gaussian processes, namely, the Girsanov Theorem and the Predictable Representation Theorem, and a major result of Malliavin-Skorohod calculus, the celebrated Clark-Haussmann-Ocone formula.

Girsanov Theorem establishes that there exists a unique probability measure $\mathbb{Q}$ equivalent to the real or empirical probability measure $\mathbb{P}$ such that

$$W_t^{\mathbb{Q}} := W_t + \frac{\mu - r}{\sigma} t, \; t \geq 0.$$

is a Brownian motion with respect to $\mathbb{Q}$. Recall that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent means that they have the same null sets.

Pricing and hedging financial derivatives VIII

Under $\mathbb{Q}$ the stochastic differential equations satisfied by S and $\tilde{S}$ are

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

and

$$d\tilde{S}_t = \sigma S_t dW_t^{\mathbb{Q}}.$$

Recall that $\tilde{G}$ is measurable with respect to $\mathcal{F}_T$ and the filtration generated by process S is the same as the filtration generated by processes $\tilde{S}$, $\tilde{X}$ or W.

The predictable representation property establishes that if $\tilde{G}$ is a functional of $L^2_{\mathbb{Q}}$, there exists a adapted process K such that

$$\tilde{G} = \mathbb{E}_{\mathbb{Q}}(\tilde{G}) + \int_0^T K_t dW_t^{\mathbb{Q}} = \mathbb{E}_{\mathbb{Q}}(\tilde{G}) + \int_0^T \frac{K_t}{\sigma \tilde{S}_t} d\tilde{S}_t \tag{7}$$

Pricing and hedging financial derivatives IX

So, combining (6) and (7) the solution of our problem is clear:

$$V_0 = \mathbb{E}_{\mathbb{Q}}(\tilde{G}) = e^{-rT}\mathbb{E}_{\mathbb{Q}}(G)$$

and

$$a_t = \frac{K_t}{\sigma \tilde{S}_t}.$$

But the weak point is how to determine process K. Malliavin-Skorohod calculus, via the Clark-Haussmann-Ocone formula, gives an elegant solution to this problem. Indeed, if $\tilde{G}$ belongs to the domain of the Malliavin derivative D, it holds

$$K_t = \mathbb{E}_{\mathbb{Q}}[D_t\tilde{G}|\mathcal{F}_t] = e^{-rT}\mathbb{E}_{\mathbb{Q}}[D_tG|\mathcal{F}_t].$$

Therefore,

$$a_t = \frac{e^{-r(T-t)}}{\sigma S_t}\mathbb{E}_{\mathbb{Q}}[D_tG|\mathcal{F}_t].$$

Pricing and hedging financial derivatives X

The more classical solution of Harrison and Pliska (before Malliavin calculus) assumes $G = g(S_T)$ and

$$V_t = \mathbb{E}[e^{-r(T-t)} g(S_T)|\mathcal{F}_t] =: F(t, S_t)$$

with

$$F(t,x) := \mathbb{E}[e^{-r(T-t)} g(x \exp\{(r - \frac{\sigma^2}{2})(T-t) + \sigma W_{T-t}\})].$$

Finally, minimizing the quadratic risk, it is easy to find

$$a_s = (\partial_x F)(s, S_s).$$

Here we are assuming the independence of increments and the fact that the derivative depends only on the final value of the underlying price.

Computation of Greeks

The second important step in the development of Malliavin-Skorohod calculus as a tool in Quantitative Finance was its application to computation of Greeks, that is, the sensitivities of financial derivative prices to variations of model parameters. Recall that these quantities are important in hedging and risk management.

Since the mathematical point of view the problem is to compute derivatives of expectations under the risk neutral measure.

Malliavin-Skorohod calculus turned out to be a useful tool to improve the numerical tractability of these computations because it allows to convert, in some sense, derivatives in products with respect to some weights.

Computation of Delta I

As a example, we can compute the Greek Δ, the derivative with respect the initial price x.

Consider a price process S under the Osborne-Samuelson model and the risk-neutral measure $\mathbb{Q}$. Denote $S_0 = x$. So, we have

$$S_t^x = x \exp\{(r - \frac{\sigma^2}{2})t + \sigma W_t\}.$$

We write S^x to emphasize the dependence on the initial condition.

Consider the price of a European type plain vanilla derivative

$$V_0 = \mathbb{E}_{\mathbb{Q}}(e^{-rT} f(S_T^x)),$$

where f is a smooth enough measurable function.

Computation of Delta II

We have, using (3) with $u(t) = \mathbf{1}_{[0,T]}(t)$, that

$$
\begin{aligned}
\partial_x V_0 &= \partial_x \mathbb{E}_{\mathbb{Q}}[e^{-rT} f(S_T^x)] \\
&= e^{-rT} \mathbb{E}_{\mathbb{Q}}[f'(S_T^x) \partial_x S_T^x] \\
&= \frac{e^{-rT}}{x} \mathbb{E}_{\mathbb{Q}}[f'(S_T^x) S_T^x] \\
&= \frac{e^{-rT}}{x} \mathbb{E}_{\mathbb{Q}}[f(S_T^x) \delta(\frac{S_T^x \mathbf{1}_{[0,T]}}{\langle DS_T^x, \mathbf{1}_{[0,T]} \rangle})] \\
&= \frac{e^{-rT}}{x} \mathbb{E}_{\mathbb{Q}}[f(S_T^x) \delta(\frac{S_T^x \mathbf{1}_{[0,T]}}{\sigma T S_T^x})] \\
&= \frac{e^{-rT}}{x \sigma T} \mathbb{E}_{\mathbb{Q}}[f(S_T^x) \delta(\mathbf{1}_{[0,T]})] \\
&= \frac{e^{-rT}}{x \sigma T} \mathbb{E}_{\mathbb{Q}}[f(S_T^x) W_T].
\end{aligned}
$$

Computation of Delta III

Note that we have used the Malliavin calculus properties $D_t g(W_T) = g'(W_T) D_t W_T$ and $D_t W_T = \mathbf{1}_{[0,T]}(t)$ for any function g smooth enough. It is clear that in the equality

$$\partial_x \mathbb{E}_{\mathbb{Q}}[e^{-rT} f(S_T^x)] = \frac{e^{-rT}}{x\sigma T} \mathbb{E}_{\mathbb{Q}}[f(S_T^x) W_T]$$

the right hand expression is much more easy to handle numerically than the left hand one. Analogously we can compute all the other Greeks.

Good references for these computations are Nualart (2006) and Kohatsu-Montero (2004). For a deep development of integration by parts formulas and applications see Bally-Caramellino (2015).

The Hull and White formula I

Fix a finite horizon $T > 0$. Consider the following slight generalization of the Osborne-Samuelson model in terms of the log price $X_t = \log S_t$:

$$X_t = x_0 + \int_0^t \alpha(s)ds + \int_0^t \sigma(s)dW_s, \ \ t \in [0, T]$$

where α and σ are, instead of constants, deterministic functions, and $x_0 := \log s_0$.
Assume the technical conditions

$$\int_0^T \sigma(s)^2 ds < \infty$$

and

$$\int_0^T (\frac{\alpha(s) - r}{\sigma(s)})^2 ds < \infty.$$

The Hull and White formula II

Therefore we can apply the Girsanov theorem and show that

$$X_t = x_0 + \int_0^t (r - \frac{1}{2}\sigma(s)^2)ds + \int_0^t \sigma(s)dW_s^{\mathbb{Q}}, \ \ t \in [0, T].$$

Note that under $\mathbb{Q}$, the random variable X_t is a normal random variable with mean

$$x_0 + rt - \frac{1}{2}\int_0^t \sigma(s)^2 ds$$

and variance

$$\int_0^t \sigma(s)^2 ds,$$

and

$$X_T = X_t + r(T-t) - \frac{1}{2}\int_t^T \sigma(s)^2 ds + \int_t^T \sigma(s)dW_s^{\mathbb{Q}}, \ t \in [0, T].$$

The Hull and White formula III

Denote

$$v(t,T) := \int_t^T \sigma(s)^2 ds$$

a quantity that is called integrated variance.
The price of a plain vanilla Call at t is given by

$$\begin{aligned} V_t &= \mathbb{E}_Q[e^{-r(T-t)}(e^{X_T} - K)^+ | \mathcal{F}_t] \\ &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(e^{X_t + r(T-t) - \frac{v(t,T)}{2}} e^{\int_t^T \sigma(s) dW_s^{\mathbb{Q}}} - K)^+ | \mathcal{F}_t] \\ &= \int_{\mathbb{R}} e^{-r(T-t)}(e^{X_t + r(T-t) - \frac{v(t,T)}{2}} e^y - K)^+ \frac{1}{\sqrt{2\pi v(t,T)}} e^{-\frac{y^2}{2v(t,T)}} dy \\ &= \int_{y > \log K - X_t - r(T-t) + \frac{v(t,T)}{2}} (e^{X_t - \frac{v(t,T)}{2}} e^y - K e^{-r(T-t)}) \frac{1}{\sqrt{2\pi v(t,T)}} e^{-\frac{y^2}{2v(t,T)}} dy \end{aligned}$$

The Hull and White formula IV

Doing the change of variable

$$z = \frac{y}{\sqrt{v(t,T)}}$$

we have

$$V_t = \int_{z > \frac{\log K - X_t - r(T-t)}{\sqrt{v(t,T)}} + \frac{\sqrt{v(t,T)}}{2}} \left(e^{X_t - \frac{v(t,T)}{2}} e^{\sqrt{v(t,T)} z} - K e^{-r(T-t)} \right) \phi(z) dz$$

where ϕ is the standard normal density.
Therefore,

$$\begin{aligned} V_t &= \int_{z > \frac{\log K - X_t - r(T-t)}{\sqrt{v(t,T)}} + \frac{\sqrt{v(t,T)}}{2}} e^{X_t - \frac{v(t,T)}{2}} e^{\sqrt{v(t,T)} z} \phi(z) dz \\ &- K e^{-r(T-t)} \int_{z > \frac{\log K - X_t - r(T-t)}{\sqrt{v(t,T)}} + \frac{\sqrt{v(t,T)}}{2}} \phi(z) dz \end{aligned}$$

The Hull and White formula V

The second integral of the right hand side is

$$Ke^{-r(T-t)}(1-\Phi(\frac{\log K - X_t - r(T-t)}{\sqrt{v(t,T)}}+\frac{\sqrt{v(t,T)}}{2}))$$

where Φ is the cumulative probability function of the standard normal law.
In relation with the first integral, using

$$(z-\sqrt{v(t,T)})^2 = z^2 + v(t,T) - 2\sqrt{v(t,T)}z$$

we have

$$\sqrt{v(t,T)}z - \frac{z^2}{2} = \frac{v(t,T)}{2} - \frac{(z-\sqrt{v(t,T)})^2}{2}$$

and therefore, the first integral becomes

$$e^{X_t}\int_{z>\frac{\log K - X_t - r(T-t)}{\sqrt{v(t,T)}}+\frac{\sqrt{v(t,T)}}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sqrt{v(t,T)})^2}{2}} dz.$$

The Hull and White formula VI

And doing the change of variable $u = z - \sqrt{v(t,T)}$ we obtain

$$e^{X_t} \int_{u > \frac{\log K - X_t - r(T-t)}{\sqrt{v(t,T)}} - \frac{\sqrt{v(t,T)}}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

Therefore,

$$\begin{aligned} V_t &= e^{X_t}\left(1 - \Phi\left(\frac{\log K - X_t - r(T-t)}{\sqrt{v(t,T)}} - \frac{\sqrt{v(t,T)}}{2}\right)\right) \\ &- Ke^{-r(T-t)}\left(1 - \Phi\left(\frac{\log K - X_t - r(T-t)}{\sqrt{v(t,T)}} + \frac{\sqrt{v(t,T)}}{2}\right)\right). \end{aligned}$$

The Hull and White formula VII

Using the fact that $\Phi(x) = 1 - \Phi(-x)$ we can write

$$\begin{aligned} V_t &= e^{X_t}\Phi\Big(\frac{X_t - \log K + r(T-t)}{\sqrt{v(t,T)}} + \frac{\sqrt{v(t,T)}}{2}\Big) \\ &- Ke^{-r(T-t)}\Phi\Big(\frac{X_t - \log K + r(T-t)}{\sqrt{v(t,T)}} - \frac{\sqrt{v(t,T)}}{2}\Big). \end{aligned}$$

The Hull and White formula VIII

Recall that the well-known Black-Scholes formula is given by $BS(t, X_t, \sigma^2)$ where X_t is the current log-price, σ^2 is the constant variance in the Osborne-Samuelson model, and

$$\begin{aligned} BS(t,x,y) &= x\Phi\left(\frac{\log x - \log K + r(T-t)}{\sqrt{y(T-t)}} + \frac{\sqrt{y(T-t)}}{2}\right) \\ &- Ke^{-r(T-t)}\Phi\left(\frac{\log x - \log K + r(T-t)}{\sqrt{y(T-t)}} - \frac{\sqrt{y(T-t)}}{2}\right). \end{aligned}$$

Note that we have shown that the price of a plain vanilla call under a Black-Scholes model with time-dependent parameters is equivalent to the Black-Scholes price but changing the constant variance by the mean of the future variances

$$\frac{1}{T-t}v(t,T) = \frac{1}{T-t}\int_t^T \sigma(s)^2 ds.$$

We can write $V_t = BS(t, X_t, \frac{v(t,T)}{T-t})$.

The Hull and White formula IX

If we assume that σ^2 is a stochastic process independent of the price process it can be shown that

$$V_t = \mathbb{E}[BS(t, X_t, \frac{v(t,T)}{T-t})|\mathcal{F}_t]$$

where now $\frac{v(t,T)}{T-t}$ is a non-adapted stochastic process because depends on the future evolutions of σ^2, see Fouque-Papanicolaou-Sircar (2000). Here we assume the natural filtration generated by processes X and σ^2. This is the so-called Hull and White formula.

To assume σ^2 is a stochastic process makes the new stochastic volatility model able to describe the volatility smile phenomenon (see Renault and Touzi (1996)).

The Hull and White formula XI

This formula can be extended to the case where correlation between prices and volatilities is allowed and also to more complicated price models as stochastic volatility jump-diffusion models.

As an example, in Alòs (2006), the previous HW formula is extended to the correlated case. That is, assume σ is $\mathbb{F}^W-$adapted and the log-price process is given by

$$X_t = x_0 + \int_0^t (r - \frac{\sigma_s^2}{2})ds + \int_0^t \sigma_s \left(\rho dW_s + \sqrt{1-\rho^2} dB_s\right).$$

The Hull and White formula X

Then, using Malliavin calculus techniques we can prove

$$
\begin{aligned}
V_t &= \mathbb{E}\left[BS(t, X_t, \frac{v(t,T)}{T-t})|\mathcal{F}_t\right] \\
&+ \frac{\rho}{2}\mathbb{E}\left[\int_t^T e^{-r(u-t)}(\partial_x^3 - \partial_x^2)BS(u, X_u, \frac{v(u,T)}{T-u})\left(\int_u^T D_u^W \sigma_r^2 dr\right)\sigma_u du|\mathcal{F}_t\right].
\end{aligned}
$$

Note that we have shown that the relevant quantity in pricing is not the variance but the mean of future variances, a non-adapted quantity. This suggest the utility of Malliavin calculus, a tool for manage non-adapted quantities.