LECTURE2: MALLIAVIN-SKOROHOD CALCULUS WITHOUT PROBABILITY

Josep Vives

Departament de Matemàtica Econòmica, Financera i Actuarial

UNIVERSITAT DE BARCELONA

2024, May 27-31

Abstract

In this lecture we introduce the abstract structure of Fock space and develop an abstract Malliavin-Skorohod calculus on this space.

This theory includes the skeleton of the concept of chaos expansion and so it can be applied to any type of processes that enjoy, in some sense, the so called chaotic representation property.

We can name it a Malliavin-Skorohod calculus without probabilities. The reference of this lecture is Nualart-Vives (1990).

Tensorial power of a Hilbert space

Let H be a real separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$ and norm $|| \cdot ||_H$.

Recall H is separable if it has a countable dense subset, and this is equivalent to assume that it has a countable orthonormal basis.

Here, we typically assume that $H = L^2([0, \infty))$ with $[0, \infty)$ endowed with the Lebesgue measure.

For any $n \geq 1$, we consider its $n-$tensorial power $H^{\otimes n}$, that is also a Hilbert space, and denote by $\langle \cdot, \cdot \rangle_{H^{\otimes n}}$ its induced scalar product.

Symmetrization

Let $\mathcal{S}_n$ be the set of permutations of $\{1, 2, \ldots, n\}$. Any permutation $\sigma \in \mathcal{S}_n$ induces an automorphism U_σ on $H^{\otimes n}$ given by $U_\sigma(x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$. Given an element $x_1 \otimes \cdots \otimes x_n$ of $H^{\otimes n}$ we can consider its symmetrization

$$x_1 \odot \cdots \odot x_n := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Consider the symmetric elements of $H^{\otimes n}$, that is, the elements that are invariant by the automorphism U_σ for any $\sigma \in \mathcal{S}$. We can introduce on this subset the modified scalar product

$$\langle x_1 \odot \cdots \odot x_n, y_1 \odot \cdots \odot y_n \rangle_{H^{\odot n}} := n! \langle x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n \rangle_{H^{\otimes n}}$$

and denote by $H^{\odot n}$ the corresponding Hilbert space. Naturally, $H^{\odot 1} = H$ and we define $H^{\odot 0} := \mathbb{R}$.

Definition of Fock space

Definition: The Fock space associated to H is the Hilbert space

$$\Phi(H) := \bigoplus_{n=0}^{\infty} H^{\odot n},$$

equipped with the scalar product

$$\langle h, g \rangle_{\Phi(H)} := \sum_{n=0}^{\infty} \langle h_n, g_n \rangle_{H^{\odot n}}$$

where $h := \sum_{n=0}^{\infty} h_n$ and $g := \sum_{n=0}^{\infty} g_n$.

Fock space associated to a measure space

From now on we will assume that $H := L^2(T)$ where $(T, \mathcal{B}, \lambda)$ is a separable measure space with λ a $\sigma-$ finite atomless measure and $\mathcal{B}$ is the Borel $\sigma-$field associated to T. Then,

$$H^{\otimes n} = L^2(T^n)$$

and $H^{\odot n}$ is the space $L_s^2(T^n)$ of square-integrable and symmetric functions on T with the modified norm

$$||f||^2_{L_s^2(T^n)} := n!||f||^2_{L^2(T^n)}.$$

Note that elements of $\Phi(L^2(T, \mathcal{B}, \lambda))$ can be written as $F = \sum_{n=0}^{\infty} f_n$ where $f_n \in L_s^2(T^n)$, for any $n \geq 1$ and $f_0 \in \mathbb{R}$.

Abstract measurability

We can introduce in the context of Fock spaces the following abstract definition of measurability with respect to a set:

Definition: Let $F \in \Phi(L^2(T, \mathcal{B}, \lambda))$ and $A \in \mathcal{B}$. We say that F is $A-$measurable if, for any $n \geq 1$, we have $f_n(t_1, \dots t_n) = 0$, $\lambda^n - a.e.$ unless $t_i \in A$, for all $i = 1, \dots, n$.

Abstract conditional expectation

Given a set, we can define an abstract concept of conditional expectation:

Definition: Given $F \in \Phi(L^2(T, \mathcal{B}, \lambda))$ we define

$$E[F|A] := \sum_{n=0}^{\infty} f_n(t_1, \ldots t_n) \mathbf{1}_A(t_1) \cdots \mathbf{1}_A(t_n).$$

Note that $E[F|\emptyset] = f_0$ and $E[F|T] = F$. So, we can define also $\mathbb{E}(F) := E[F|\emptyset]$.

Abstract predictability

Finally we can introduce an abstract concept of predictability:

Definition: Given an element $u \in L^2(T, \Phi(H))$ and a set $A \in \mathcal{B}$, we say that u is an elementary predictable process if

$$u(t) = F \otimes \mathbf{1}_{A^c}(t)$$

and F is $A-$measurable.

The annihilation operator

We introduce now an abstract operator, usually called annihilation operator in the context of Fock space, that will be the abstract version of the so called Malliavin derivative in different probabilistic contexts.

Definition: Consider $F \in \Phi(L^2(T))$. We define DF as an element of $L^2(T, \Phi(L^2(T)))$ such that

$$D_t F := \sum_{n=1}^{\infty} n f_n(\cdot, t), \; t - a.e.$$

provided the sum converges in $L^2(T, \Phi(L^2(T)))$.

Domain of the annihilation operator

Operator DF exists if

$$\begin{aligned} ||DF||^2_{L^2(T,\Phi(L^2(T)))} &= \int_T ||D_tF||^2_{\Phi(L^2(T))}\lambda(dt) \\ &= \sum_{n=1}^{\infty} n^2(n-1)! \int_T ||f_n(\cdot,t)||^2_{L^2(T^{n-1})}\lambda(dt) \\ &= \sum_{n=1}^{\infty} nn! ||f_n||^2_{L^2(T^n)} < \infty. \end{aligned}$$

We define the domain of the operator D as

$$DomD = \{F \in \Phi(L^2(T)) : ||DF||^2_{L^2(T,\Phi(L^2(T)))} = \sum_{n=1}^{\infty} nn! ||f_n||^2_{L^2(T^n)} < \infty\} \subseteq \Phi(L^2(T)).$$

Denseness of the operator annihilation operator

The operator D is an unbounded and closed operator with a dense domain $DomD$.

$DomD$ is dense. If $F \in \Phi(L^2(T))$ has a finite expansion, that is, $F = \sum_{n=0}^{m} f_n$, it is clear that it belongs to $DomD$ because $||DF||^2_{L^2(T,\Phi(L^2(T)))}$ is a finite sum. On other hand, by definition, any element of $\Phi(L^2(T))$ can be approximated by its finite partial sums, and so, finite expansions are dense in $\Phi(L^2(T))$.

Closeability of the annihilation operator I

The operator D is closeable. Let $F^{(N)}$ a sequence of elements of $DomD$ converging to $F \in \Phi(L^2(T))$. Assume on other hand that $DF^{(N)}$ converges to $u \in L^2(T, \Phi(L^2(T)))$. Then we can show that $F \in DomD$ and $DF = u$. Indeed, from the two hypothesis we have that, for every $n \geq 0$,

$$||f_n^{(N)} - f_n||_{L^2(T^n)} \longrightarrow 0$$

and

$$||nf_n^{(N)}(\cdot, t) - u_{n-1}(\cdot, t)||_{L^2(T^n)} \longrightarrow 0,$$

when $N \uparrow \infty$. Note that

$$u_t = \sum_{n=0}^{\infty} u_n(\cdot, t), \forall t \in T.$$

Closeability of the annihilation operator II

Denote by $u^{(m)}$ and F_m the series representing u and F cut at m. Consider also the corresponding functionals $F_m^{(N)}$. Being F_m a finite sum, it is clear that it belongs to $DomD$. And on other hand,

$$||u^{(m)} - DF_m||_{L^2(T,\Phi(H))} \leq ||u^{(m)} - DF_m^{(N)}||_{L^2(T,\Phi(H))} + ||DF_m^{(N)} - DF_m||_{L^2(T,\Phi(H))}.$$

In relation with the first term on the right hand side we have

$$||u^{(m)} - DF_m^{(N)}||_{L^2(T,\Phi(H))} \leq ||u - DF^{(N)}||_{L^2(T,\Phi(H))}$$

and taking $N \uparrow \infty$ it converges to 0 by hypothesis.

Closeability of the annihilation operator III

In relation with the second term we have

$$||DF_m^{(N)} - DF_m||_{L^2(T,\Phi(H))} = \sum_{n=1}^{m} nn!||f_n^{(N)} - f_n||_{L^2(T^n)}$$

that converges also to 0 because is a finite sum of terms converging to 0.
So we have proved that $u^{(m)} = DF_m$, for any $m \geq 1$ and, in particular, that for any $n \geq 1$, we have

$$u_{n-1}(\cdot) = nf_n(\cdot, t) \quad \lambda^n - a.e.$$

Being $u \in L^2(T, \Phi(L^2(T)))$ this implies that $F \in DomD$ and $u = DF$.

Extension of the annihilation operator

We can extend the previous definition to an unbounded and closed operator from $\Phi(L^2(T))$ to $\Phi(L^2(T))$:

Definition: Given $h \in L^2(T)$ we can define

$$D_h F := \sum_{n=0}^{\infty} n \int_T f_n(\cdot, t) h(t) \lambda(dt)$$

provided that the series converges in $\Phi(L^2(T))$.

Note that the domain of D_h, denoted by $DomD_h$, is the set of elements $F \in \Phi(L^2(T))$ such that

$$\sum_{n=1}^{\infty} nn! \int_T ||f_n(\cdot, t) h(t)||^2_{L^2(T^{n-1})} \lambda(dt) < \infty.$$

By Schwarz inequality it is clear that $DomD \subseteq DomD_h$.

Annihilation operator and conditional expectation

In relation with the notion of measurability with respect a Borel set A introduced before we have the following result:

Proposition: For any $F \in DomD$ and $A \in \mathcal{B}$, we have

$$D_t E[F|A] = E[D_t F|A] \cdot \mathbf{1}_A(t)$$

The proof is immediate using the definition of measurability and writing explicitly both sides of the equality.

An immediate consequence of this result is that if $F \in DomD$ is $A-$measurable, then $D_t F = 0$, for all $t \in A^c$, $a.e.$

Iterated annihilation operator

We can iterate the definition of annihilation operator in a natural way.

Definition: For any $k \geq 1$, we can define D^k as an operator from $\Phi(L^2(T))$ to $L^2_s(T^k, \Phi(L^2(T)))$ such that

$$D^k_{t_1,\dots,t_k} F := \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) f_n(\cdot, t_1, \dots, t_k), \ \ \lambda^k - a.e.$$

provided the sum converges in $L^2(T^k, \Phi(L^2(T)))$. We denote the domain of operator D^k as $DomD^k$.

It is immediate to see that $F = \sum_{n=0}^{\infty} f_n \in DomD^k$ if and only if

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) n! ||f_n||^2_{L^2(T^n)} < \infty.$$

The creation operator I

Given the Hilbert space $L^2(T,\Phi(L^2(T)))$, we can introduce, for any $n\geq 0$, the space $\hat{L}^2(T^{n+1})$ as the subspace of $L^2(T^{n+1})$ of all square integrable functions on T^{n+1} such that they are symmetric in the first n variables. Then, for any $u\in L^2(T,\Phi(L^2(T)))$, we can write

$$u=\sum_{n=0}^{\infty}u_n$$

where $u_n\in\hat{L}^2(T^{n+1})$ and

$$||u||^2_{L^2(T,\Phi(L^2(T)))}=\int_T\sum_{n=0}^{\infty}n!||u_n(\cdot,t)||^2_{L^2(T^n)}\lambda(dt)=\sum_{n=0}^{\infty}n!||u_n||^2_{L^2(T^{n+1})}.$$

Moreover, if $u,v\in L^2(T,\Phi(L^2(T)))$, we can define

$$\langle u,v\rangle_{L^2(T,\Phi(L^2(T)))}=\sum_{n=0}^{\infty}n!\langle u_n,v_n\rangle_{L^2(T^{n+1})}.$$

The creation operator II

The following is the definition of the creation operator:

Definition: Given $u \in L^2(T, \Phi(L^2(T)))$, we define

$$\delta(u) := \sum_{n=0}^{\infty} \tilde{u}_n$$

where $\tilde{u}_n$ denotes the symmetrization of u_n with respect its $n+1$ variables, provided the series converges, that is

$$\sum_{n=0}^{\infty} (n+1)! ||\tilde{u}_n||^2_{L^2(T^n)} < \infty.$$

We will denote by $Dom\delta$ the domain of the operator δ into $L^2(T, \Phi(L^2(T)))$.

THE CREATION OPERATOR AS EXTENSION OF ITÔ INTEGRAL

The following results characterize the action of operator δ over elementary predictable processes and suggest the idea that the operator δ could be an extension of the Itô stochastic integral.

Proposition: Consider an elementary predictable process $u = F \otimes \mathbf{1}_{A^c}$, where $F = \sum_{n=0}^{\infty} f_n$ and $A \in \mathcal{B}$ such that $\lambda(A^c) < \infty$. Then, $u \in Dom\delta$, and

$$\delta(u) = \sum_{n=0}^{\infty} f_n \odot \mathbf{1}_{A^c}$$

PROOF

The result is immediate from the definition of δ. The series converges because

$$
\begin{aligned}
||\delta(u)||_{\Phi(L^2(T))} &= \sum_{n=0}^{\infty}(n+1)!||f_n \odot \mathbf{1}_{A^c}||^2_{L^2(T^{n+1})} \\
&= \sum_{n=0}^{\infty} n!||f_n \otimes \mathbf{1}_{A^c}||^2_{L^2(T^{n+1})} \\
&= ||F||^2_{\Phi(L^2(T))}\lambda(A^c) < \infty.
\end{aligned}
$$

Note that

$$
(f_n \odot \mathbf{1}_{A^c})^2(t_1, \ldots, t_n, t) = \frac{1}{n+1} f_n^2(t_1, \ldots, t_n)\mathbf{1}_{A^c}(t).
$$

Duality

An auxiliary lemma that is useful in proofs is the following:

Lemma: If $f, g \in L^2(T^n)$, we have

$$\int_{T^n} f \cdot \tilde{g}\, \lambda(dt_1) \cdots \lambda(dt_n) = \int_{T^n} \tilde{f} \cdot \tilde{g}\, \lambda(dt_1) \cdots \lambda(dt_n)$$

Operators D and δ defined above satisfy the following duality relationship. This is the basic key fact of Malliavin-Skorohod calculus.

Theorem If $F \in DomD$ and $u \in Dom\delta$, then

$$\langle u, DF \rangle_{L^2(T,\Phi(H))} = \langle F, \delta(u) \rangle_{\Phi(L^2(T))}.$$

Proof

Consider $u = \sum_{n=0}^{\infty} u_n$ and $F = \sum_{n=0}^{\infty} f_n$. Recall that, for any $n \geq 0$, f_n is a symmetric function. Then,

$$
\begin{aligned}
\langle u, DF \rangle_{L^2(T,\Phi(L^2(T)))} &= \int_T \langle u(\cdot, t), D_t F \rangle_{\Phi(L^2(T))} \lambda(dt) \\
&= \sum_{n=0}^{\infty} n! \int_T \langle u_n(\cdot, t), (n+1) f_{n+1}(\cdot, t) \rangle_{L^2(T^n)} \lambda(dt) \\
&= \sum_{n=0}^{\infty} (n+1)! \int_{T^{n+1}} u_n(\cdot, t) f_{n+1}(\cdot, t) \lambda(dt_1) \cdots \lambda(dt_n) \lambda(dt) \\
&= \int_{T^{n+1}} \sum_{n=0}^{\infty} (n+1)! \, \tilde{u}_n(\cdot, t) f_{n+1}(\cdot, t) \lambda(dt_1) \cdots \lambda(dt_n) \lambda(dt) \\
&= \langle F, \delta(u) \rangle_{\Phi(L^2(T))}.
\end{aligned}
$$

Closeability and denseness of the creation operator

This duality relationship says that δ is the adjoint operator of D. We can say that $u \in Dom\delta$ if there exists an element of $\Phi(H)$, defined as $\delta(u)$, such that the duality relationship stands for any $F \in DomD$. Then we have the following result:

Corollary: The operator δ is a closed operator densely defined on $L^2(T, \Phi(L^2(T)))$.

Proof: The fact that D is densely defined and closed implies that its adjoint operator δ is also densely defined and closed. This is a well-known result of the theory of unbounded operators; see for example Balakrishnan (1976).

The space $\mathcal{L}^2$

We introduce now a subspace of $Dom\delta$ that will allow us to have a more structured domain of operator δ and proof new results:

Definition: We define the space $\mathcal{L}^2$ as the space of elements in $L^2(T, \Phi(L^2(T)))$ such that $u_t \in DomD$, for all t, a.e., and $Du \in L^2(T^2, \Phi(L^2(T)))$.

The following lemma is immediate and characterizes the space $\mathcal{L}^2$:

Lemma: $u \in \mathcal{L}^2$ if and only if

$$\sum_{n=1}^{\infty} n(n+1)! ||\tilde{u}_n||^2_{L^2(T^{n+1})} < \infty$$

and so

$$\mathcal{L}^2 \subseteq Dom\delta.$$

Scalar product of creation operators

With elements in $\mathcal{L}^2$ we can establish new relationships between operators D and δ.

Proposition Let u and v elements of $\mathcal{L}^2$. Then, we have

$$\langle \delta(u), \delta(v) \rangle_{\Phi(L^2(T))} = \langle u, v \rangle_{L^2(T,\Phi(H))} + \int_T \int_T \langle D_s u_t, D_t v_s \rangle_{\Phi(L^2(T))} \lambda(ds)\lambda(dt).$$

Proof I

On one hand

$$\langle \delta(u), \delta(v) \rangle_{\Phi(L^2(T))} = \sum_{n=0}^{\infty} (n+1)! \int_{T^{n+1}} \tilde{u}_n(\cdot, t) \tilde{v}_n(\cdot, t) \lambda(dt_1) \cdots \lambda(dt_n) \lambda(dt).$$

On other hand,

$$\begin{aligned} \langle u, v \rangle_{L^2(T, \Phi(L^2(T)))} &= \int_T \langle u_t, v_t \rangle_{\Phi(L^2(T))} \lambda(dt) \\ &= \sum_{n=0}^{\infty} n! \int_{T^{n+1}} u_n(\cdot, t) v_n(\cdot, t) \lambda(dt_1) \cdots \lambda(dt_n) \lambda(dt). \end{aligned}$$

Proof II

The difference between the two previous terms is

$$\sum_{n=0}^{\infty} n! \int_{T^{n+1}} [(n+1)\tilde{u}_n(\cdot,t)\tilde{v}_n(\cdot,t) - u_n(\cdot,t)v_n(\cdot,t)]\lambda(dt_1)\cdots\lambda(dt_n)\lambda(dt)$$

$$= \sum_{n=0}^{\infty} n! \int_{T^{n+1}} u_n(\cdot,t)[(n+1)\tilde{v}_n(\cdot,t) - v_n(\cdot,t)]\lambda(dt_1)\cdots\lambda(dt_n)\lambda(dt)$$

$$= \sum_{n=0}^{\infty} n! \sum_{i=0}^{n} \int_{T^{n+1}} u_n(\cdot,t)v_n(\cdot,t_i)\lambda(dt_1)\cdots\lambda(dt_n)\lambda(dt).$$

Proof III

Now, using that v_n is symmetric with respect the first n variables and putting $s := t_n$ we can write the last term as

$$\sum_{n=0}^{\infty} n n! \int_{T^{n+1}} u_n(\cdot, s, t) v_n(\cdot, t, s) \lambda(dt_1) \cdots \lambda(dt_{n-1}) \lambda(ds) \lambda(dt)$$

$$= \int_{T^2} \langle D_s u_t, D_t v_s \rangle_{\Phi(L^2(T))} \lambda(ds) \lambda(dt).$$

Composition of operators

Next two propositions clarify the relations between operators D and δ

Proposition Let $u \in \mathcal{L}^2$ and assume $D_t u \in Dom\delta$ and $\delta(D_t u) \in L^2(T, \Phi(L^2(T)))$. Then $\delta(u) \in DomD$ and

$$D_t\delta(u) = u_t + \delta(D_t u).$$

Proof: Let $u = \sum_{n=1}^{\infty} u_n$. Then $\delta(u) = \sum_{n=1}^{\infty} \tilde{u}_n$ and

$$D_t\delta(u) = \sum_{n=1}^{\infty}(n+1)u_n(\cdot, t) = \sum_{n=1}^{\infty} u_n(\cdot, t) + \sum_{n=1}^{\infty} nu_n(\cdot, t) = u_t + \delta(D_t u).$$

Ornstein-Uhlenbeck operator

Definition: Given $F \in \Phi(H)$, we define the so-called Ornstein-Uhlenbeck operator

$$LF := \sum_{n=0}^{\infty} n f_n,$$

provided

$$\sum_{n=1}^{\infty} n^2 n! ||f_n||^2_{L^2(T^n)} < \infty.$$

Denote by $DomL$ its domain. Note that $DomL = DomD^2 \subseteq DomD$.

Operator L

It is interesting consider also the operator

$$L^{-1}F := \sum_{n=1}^{\infty} \frac{1}{n} f_n,$$

that allows to write $L^{-1}LF = LL^{-1}F = F$, for any F such that $f_0 = 0$.

Proposition: If $F \in DomL$, we have $\delta(DF) = LF$.

Proof: The proof is immediate by applying the definitions of the operators D, δ and L.

Remark: For any F such that $f_0 = 0$, we can write $F = \delta(DL^{-1}F)$.

This is the basis of the combination of Malliavin calculus and Stein method. See Nourdin-Peccati (2012).

ABSTRACT CHO FORMULA

The following result is an abstract and non probabilistic version of the so-called Clark-Haussman-Ocone formula (CHO formula).

Theorem Let $F \in DomD \subseteq \Phi(L^2(T))$. Then,

$$F = \mathbb{E}F + \delta(E[D_t F | [0, t]]).$$

Proof I

We have

$$F = \sum_{n=0}^{\infty} f_n, \quad , f_n \in L_s^2(T^n)$$

and

$$D_t F = \sum_{n=1}^{\infty} n f_n(\cdot, t).$$

Then,

$$E[D_t F | [0, t]] = \sum_{n=1}^{\infty} n f_n(t_1, \ldots, t_{n-1}, t) \prod_{i=1}^{n-1} \mathbf{1}_{[0,t]}(t_i).$$

Proof II

Symmetrizing we obtain

$$\delta(E[D_t F|[0,t]]) = \sum_{n=1}^{\infty} f_n(t_1, \ldots, t_{n-1}, t_n) = F - \mathbb{E}(F).$$

Observe that f_n is symmetric with respect all its n variables and the product of indicators is symmetric with respect the first $n-1$ variables, so it has to be symmetrized only with respect to t, and t can be in n different positions with respect $t_1, \ldots, t_{n-1}$. So, the symmetrization of

$$\prod_{i=1}^{n-1} \mathbf{1}_{[0,t]}(t_i)$$

is equal to

$$\frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\{t_1 < \cdots t_{j-1} < t_n < t_j < \cdots t_{n-1}\}} = \frac{1}{n}.$$