Lecture 3: Infinite divisible distributions, Lévy processes and additive processes

Josep Vives

Departament de Matemàtica Econòmica, Financera i Actuarial

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Abstract

The purpose of this lecture is to double.

On one hand, to recall the notion of infinite divisibility and the Lévy-Khinchin representation of infinite divisible probability measures.

And on other to introduce the central notions of Lévy and additive processes and present the celebrated Lévy-Itô decomposition of Lévy and additive processes. Recall that a Lévy process is a particular case of additive process.

In relation of advanced Probability Theory we refer the audience, for example, to the books by Dudley (2002), Kallenberg (2002) or Klenke (2020), among many others. And in relation with Lévy (and additive) processes, we refer the audience to the books by Protter (2005), Cont and Tankov (2004), Applebaum (2009) and Sato (Second edition 2013) among others.

Measures

Consider $(E, \mathcal{E}, \mu)$ a complete measure space. Measure μ is a positive measure. If $\mu(E)$ is finite we say that μ is a finite measure.

In this course we are specially interested in the case $E = [0, \infty) \times \mathbb{R}$ endowed with its Borel $\sigma-$algebra. Note that E is a subspace of $\mathbb{R}^2$.

Sometimes we restrict $[0, \infty)$ to $[0, T]$ and $\mathbb{R}$ to $\mathbb{R}_0 := \mathbb{R} - \{0\}$.

We denote by $\mathcal{B}$ the Borel $\sigma-$algebra of $\mathbb{R}$ and $\mathcal{B}_0$ the Borel $\sigma-$algebra of $\mathbb{R}_0$.

We say that a positive measure μ on $(E, \mathcal{E})$ is $\sigma-$finite if we can write

$$E = \bigcup_{i=1}^{\infty} E_i$$

with $E_i \in \mathcal{E}$ and $\mu(E_i) < \infty$, for all $i \geq 1$. Of course, finite measures are $\sigma-$finite.

Absolute continuous measures / Diffuse measures

Consider the Lebesgue measure on $\mathbb{R}^d$. For any locally integrable function $g : \mathbb{R}^d \longrightarrow [0, \infty)$, we can define a positive measure associated with g by

$$\mu(A) = \int_A g(x)dx. \tag{1}$$

Function g is called the density of measure μ with respect the Lebesgue measure, or simply the density of μ.

Conversely, a measure μ is called *absolutely continuous* if there exists a positive measurable function g such that previous equality stands.

A measure that gives zero mass to any point of E is said to be diffuse (or nonatomic or atomless). Examples of diffuse measures are Lebesgue measure and any measure with a density.

Counting measures

Another important type of measure is the Dirac delta measure δ_x, for a given $x \in E$. Recall that, for any $A \in \mathcal{E}$, $\delta_x(A)$ is equal to 1 if $x \in A$ and equal to 0 if $x \notin A$. In other words, $\delta_x(A) = \mathbf{1}_A(x)$.

This measure can be extended immediately to a finite or countable sum of Dirac delta measures. Given $\{x_i, i \geq 1\} \subseteq E$, we can consider

$$\mu := \sum_i \delta_{x_i}.$$

In this case, we say that μ is a counting measure because given $A \in \mathcal{E}$, $\mu(A)$ counts the number of elements of the set $\{x_i, i \geq 1\}$ that are in A. Note that in particular it is an integer valued measure, that is, it takes values in $\mathbb{N} \cup \{\infty\}$. Obviously, counting measures are integer valued measures.

Lévy measures I

We say that a positive measure ν on $\mathbb{R}$ is a Lévy measure if it satisfies the following two conditions:

a) $\nu(\{0\}) = 0$

b) $\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty$.

For any Lévy measure ν on $\mathbb{R}$ the following properties are straightforward.

a) For any $\epsilon > 0$, measure ν is finite on $(-\epsilon, \epsilon)^c$

b) For any $\epsilon > 0$, $\int_{-\epsilon}^{\epsilon} x^2 \nu(dx) < \infty$.

c) Measure ν is $\sigma-$finite.

Lévy measures II

All finite measures null at the origin are Lévy measures. If ν is not null at the origin we can define $\nu_0 := \nu - \nu(\{0\})$, and measure ν_0 is a Lévy measure.

But many infinite measures are also Lévy measures. For example, $\nu(dx) = \frac{dx}{x^2}, x \neq 0$.

Characteristic functions

Given a probability measure we define its characteristic function (or Fourier transform) φ_μ, or φ, as the complex-valued function given by

$$\varphi_\mu(t) := \int_{\mathbb{R}} e^{itx} \mu(dx).$$

In the language of random variables we have, given a random variable X,

$$\varphi_X(t) = \mathbb{E}(e^{itX}).$$

Thanks to the inversion theorem of the Fourier transform, characteristic functions determine the corresponding probability measures, in the sense that if $\varphi_\mu = \varphi_\nu$ then, $\mu = \nu$.

Theorem of continuity of Lévy

Theorem: Let $\{\mu_n, n \geq 1\}$ be a sequence of probability measures with corresponding characteristic functions $\{\varphi_n, n \geq 1\}$. If φ_n converge pointwise to a function φ, and φ is continuous at 0, then, there exists μ such that $\varphi = \varphi_\mu$ and μ is the weak limit of $\{\mu_n, n \geq 1\}$.

Convolution of probability measures

Another central notion is the notion of convolution of probability measures that is the counterpart of the idea of sum of random variables.

Given two probability measures μ_1 and μ_2 we define its convolution as

$$(\mu_1 * \mu_2)(B) := \int_{\mathbb{R}} \mu_2(B-x)\mu_1(dx) = \int_{\mathbb{R}} \mu_1(B-y)\mu_2(dy).$$

It is straightforward to see that the convolution of two probability measures is a new probability measure.

For any probability measure μ and for any $a \in \mathbb{R}$, $(\mu * \delta_a)(B) = \mu(B-a)$ and in particular, $(\mu * \delta_0)(B) = \mu(B)$. So, δ_0 is the neutral element of convolution. In particular, given $b \in \mathbb{R}$, $\delta_a * \delta_b = \delta_{a+b}$.

Convolution of finite measures

Given $\lambda \in \mathbb{R}$, and μ and ν probability measures, we have $(\lambda\mu) * \nu = \lambda(\mu * \nu)$.

This allows to extend the definition of convolution to finite measures. In fact, given two finite measures $\bar{\mu}$ and $\bar{\nu}$ we can write for any Borel set *B*,

$$(\bar{\mu} * \bar{\nu})(B) := \bar{\mu}(\mathbb{R})\bar{\nu}(\mathbb{R})(\frac{\bar{\mu}}{\bar{\mu}(\mathbb{R})} * \frac{\bar{\nu}}{\bar{\nu}(\mathbb{R})})(B).$$

Convolution of random variables

Consider two independent random variables X and Y and take $B \in \mathcal{B}$. We have

$$\mathbb{P}(X + Y \in B) = (P_X * P_Y)(B).$$

and for $X_1, \ldots, X_n$ independent random variables, we have

$$\mathbb{P}(X_1 + \cdots + X_n \in B) = (P_{X_1} * \cdots * P_{X_n})(B).$$

Infinite divisibility of probability measures

We introduce now the central notion of infinite divisible probability measure.

Recall that our goal is to present the Lévy-Khinchin theorem that characterizes all infinite divisible probability measures.

To fix ideas we consider only probability measures on $\mathbb{R}$, but results can be extended quite straightforwardly to $\mathbb{R}^d$ for any $d \geq 1$.

Definition: We say that a probability measure μ on $\mathbb{R}$ is infinite divisible if, for any $n \geq 1$, there exists a probability measure μ_n such that

$$\mu = \mu_n * \cdots * \mu_n = \mu_n^{*n}.$$

Infinite divisibility of random variables

In terms of random variables, infinite divisibility is equivalent to the following definition:

Definition: If X is a random variable with probability law $\mathcal{L}(X) = \mu$, we say that it is infinite divisible if, for any $n \geq 1$, there exists a sequence $\{X_1^{(n)}, \ldots, X_n^{(n)}\}$ of n independent and identically distributed random variables, copies of a random variable $X^{(n)}$, with probability law $\mu_n = \mathcal{L}(X^{(n)})$, such that

$$\mathcal{L}(X) = \mathcal{L}(X_1^{(n)} + \cdots X_n^{(n)}).$$

Infinite divisibility and characteristic functions

In terms of characteristic functions, infinite divisibility can be characterized by the following proposition.

Proposition: A probability measure μ is infinitely divisible if, for any $n \geq 1$, there exist a probability measure μ_n such that

$$\varphi(t) = \varphi_n(t)^n,$$

for all $t \in \mathbb{R}$, where φ and φ_n are, respectively, the characteristic functions of μ and μ_n.

Uniqueness of convolution roots

The following proposition guarantees μ_n is unique for any $n \geq 1$, because $\varphi_n(z) := \varphi(z)^{\frac{1}{n}}$ is uniquely defined.

Proposition: If μ is an infinite divisible probability measure on $\mathbb{R}$ such that $\mu = \mu_n^{*n}$ and φ and φ_n are the corresponding characteristic functions we have

a) $\varphi(t) \neq 0$ for all $t \in \mathbb{R}$.
b) $\varphi_n = \varphi^{\frac{1}{n}}$ and is unique.
c) $\lim_{n\uparrow\infty} \mu_n = \delta_0$.

Examples of infinite divisible probability laws I

The following three examples play an important role in the family of infinite divisible probability laws.

Normal law: If X is a normal random variable with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ we have

$$\mathbb{E}(e^{itX}) = \int_{\mathbb{R}} e^{itx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\} dx = e^{it\mu} \int_{\mathbb{R}} e^{it\sigma z} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{z^2}{2}\} dz = e^{it\mu} e^{-\frac{\sigma^2 t^2}{2}}.$$

Then, $\exp\{i\mu t - \frac{\sigma^2}{2} t\} = (\exp\{i\frac{\mu}{n} t - \frac{\sigma^2}{2n} t\})^n$ and therefore, $N(\mu, \sigma^2) = (N(\frac{\mu}{n}, \frac{\sigma^2}{n}))^{*n}$.

Examples of infinite divisible probability laws II

Poisson law: If X is a Poisson random variable with parameter $\lambda > 0$, we have

$$\mathbb{E}(e^{itX}) = \sum_{k=0}^{\infty} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} = e^{\lambda(e^{it}-1)}.$$

Then, $e^{\lambda(e^{it}-1)} = (e^{\frac{\lambda}{n}(e^{it}-1)})^n$ and therefore, $P(\lambda) = (P(\frac{\lambda}{n}))^{*n}$.

Cauchy law: If X is a Cauchy random variable with parameter $\alpha > 0$ we have

$$\mathbb{E}(e^{itX}) = \int_{\mathbb{R}} e^{itx} \frac{\alpha}{\pi(\alpha^2 + x^2)} dx = 2 \int_0^{\infty} \frac{\alpha \cos(tx)}{\pi(\alpha^2 + x^2)} dx = e^{-\alpha|t|}.$$

Then, $e^{-\alpha|t|} = (e^{-\frac{\alpha}{n}|t|})^n$ and therefore, $C(\alpha) = (C(\frac{\alpha}{n}))^{*n}$.

The compound Poisson law with respect to a finite measure

Let μ be a finite measure on $\mathbb{R}$. We can define the following measure

$$(P\mu)(B) := e^{-\mu(\mathbb{R})} \sum_{k=0}^{\infty} \frac{1}{k!} \mu^{*k}(B), \; B \in \mathcal{B}.$$

It is straightforward to see that $P\mu$ is a probability measure because is an increasing limit of finite measures and $(P\mu)(\mathbb{R}) = 1$. This probability measure is called the compound Poisson probability measure associated to the finite measure μ.

Note in particular that if $\mu = \lambda\delta_1$, with $\lambda \geq 0$, we have that $P(\lambda\delta_1)$ is the Poisson law of parameter λ. Indeed,

$$(P(\lambda\delta_1))(B) = e^{-\lambda\delta_1(\mathbb{R})} \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k \delta_1^{*k}(B) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_k(B) = e^{-\lambda} \sum_{k \in B} \frac{\lambda^k}{k!}.$$

Interpretation of the compound Poisson law

The following lemma shows the usefulness of the compound Poisson law.

Lemma: Let $\{X_i, i \geq 1\}$ be a family of independent and identically distributed random variables with law $\frac{\mu}{\mu(\mathbb{R})}$ and N is Poisson random variable with parameter $\mu(\mathbb{R})$, independent of all X_i. We have

$$\mathcal{L}(X_1 + \cdots + X_N) = P\mu.$$

Another interesting property is the following lemma:
Lemma: Given an infinite divisible probability measure μ there exist a sequence of probability measures μ_n such that

$$\mu = \lim_{n \uparrow \infty} P(n\mu_n).$$

Proof

We have, for any Borel set B,

$$\begin{aligned}
\mathbb{P}(X_1+\cdots+X_N\in B) &= \sum_{n=0}^{\infty}\mathbb{P}(X_1+\cdots+X_n\in B|N=n)e^{-\mu(\mathbb{R})}\frac{\mu(\mathbb{R})^n}{n!}\\
&= \sum_{n=0}^{\infty}e^{-\mu(\mathbb{R})}\frac{\mu(\mathbb{R})^n}{n!}\frac{1}{\mu(\mathbb{R})^n}\mu^{*n}(B)\\
&= \sum_{n=0}^{\infty}e^{-\mu(\mathbb{R})}\frac{1}{n!}\mu^{*n}(B)\\
&= (P\mu)(B).
\end{aligned}$$

Characteristic function of a compound Poisson law I

The previous lemma allows us to compute the characteristic function of a compound Poisson law. Indeed, with the same notation of the previous lemma, we have

$$
\begin{aligned}
\varphi_{P_\mu}(t) &= \mathbb{E}(e^{it(X_1+\cdots+X_N)}) \\
&= \sum_{k=0}^{\infty} e^{-\mu(\mathbb{R})} \frac{\mu(\mathbb{R})^k}{k!} \frac{1}{(\mu(\mathbb{R}))^k} \Big(\int_{\mathbb{R}} e^{itx} \mu(dx) \Big)^k \\
&= \sum_{k=0}^{\infty} e^{-\mu(\mathbb{R})} \frac{1}{k!} \Big(\int_{\mathbb{R}} e^{itx} \mu(dx) \Big)^k \\
&= e^{-\mu(\mathbb{R})} \exp\Big\{ \int_{\mathbb{R}} e^{itx} \mu(dx) \Big\} \\
&= \exp\Big\{ \int_{\mathbb{R}} (e^{itx} - 1) \mu(dx) \Big\}
\end{aligned}
$$

Characteristic function of a compound Poisson law II

Note that if μ is a finite measure and $\lambda = \mu(\mathbb{R})$ we can define the probability measure $\bar{\mu} = \frac{\mu}{\lambda}$. Then,

$$\varphi_{P_\mu}(t) = \exp\{\lambda(\varphi_{\bar{\mu}}(t) - 1)\}.$$

The reverse is also true, that is, if φ is a characteristic function of a certain probability measure ν and $\lambda > 0$, the function

$$\exp\{\lambda(\varphi(t) - 1)\}$$

is the characteristic function of the probability measure $P(\lambda\nu)$.

Properties of the compound Poisson law I

Other properties are given by the following proposition:

Proposition: Given $\lambda \geq 0$ and μ and ν finite measures we have

a) $P(\mu * \nu) = (P\mu) * (P\nu)$.
b) $P(\lambda\nu) = (P\nu)^{*\lambda}$.
c) $P(\lambda\delta_0) = \delta_0$.
d) If $\nu^0 := \nu - \nu(\{0\})\delta_0$ we have $P\nu^0 = P\nu = (P\nu) * (P(\nu(\{0\})\delta_0))$.

Properties of the compound Poisson law II

Note that as a consequence of the previous proposition we have

$$(P\mu) = (P\frac{\mu}{n}) * \cdots * (P\frac{\mu}{n}) = (P\frac{\mu}{n})^{*n}$$

and therefore, $P\mu$ is a new example of infinite divisible law.

The following proposition will play an important role.

Proposition: Consider μ is a probability measure on $\mathbb{R}$. Then,

$$\sup_{B\in\mathcal{B}} |(P\mu)(B) - \mu(B)| \leq (\mu(\mathbb{R} - \{0\}))^2.$$

The compound Poisson law with respect a Lévy measure I

Towards the goal to prove the Lévy-Khinchin decomposition we need to extend the definition of a compound Poisson measure with respect a finite measure to a compound Poisson measure with respect to a Lévy measure.

Assume ν is a Lévy measure. We know that $\nu([-\epsilon,\epsilon]^c) < \infty$. Therefore, we can consider for any ϵ the finite measures such that for any Borel set B,

$$\nu^\epsilon(B) := \nu(B \cap [-\epsilon,\epsilon]^c).$$

On other hand, for any ϵ we can define the finite constant

$$c_\epsilon := \int_{\epsilon<|x|\le 1} x\nu(dx).$$

The compound Poisson law with respect a Lévy measure II

Then, we can consider the probability measure

$$\mu_\epsilon := \delta_{-c_\epsilon} * P\nu^\epsilon.$$

Its characteristic function is

$$\begin{aligned}\varphi_{\mu_\epsilon}(t) &= e^{-itc_\epsilon}\exp\{\int_{|x|>\epsilon}(e^{itx}-1)\nu(dx)\} \\ &= \exp\{\int_{|x|>\epsilon}(e^{itx}-1-itx\mathbf{1}_{[-1,1]}(x))\nu(dx)\}\end{aligned}$$

The compound Poisson law with respect a Lévy measure III

The following lemma allow us to define a compound Poisson measure with respect a Lévy measure.

Proposition If ν is a Lévy measure, we define

$$\varphi(t) := \exp\{\int_{\mathbb{R}} (e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x))\nu(dx)\} = \lim_{\epsilon \downarrow 0} \varphi_{\mu_\epsilon}(t)$$

and φ is the characteristic function of a probability law.

Proof I

It is straightforward to see the pointwise convergence

$$\lim_{\epsilon \downarrow 0}(e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x))\mathbf{1}_{\{|x|>\epsilon\}} = e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x).$$

On other hand, we can write

$$|e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x)| = |e^{itx} - 1 - itx|\mathbf{1}_{[-1,1]}(x) + |e^{itx} - 1|\mathbf{1}_{\{|x|>1\}}(x)$$

By Taylor expansion the first term at the right hand side is bounded by $e^{|t|}t^2x^2\mathbf{1}_{[-1,1]}(x)$ and the second term is bounded by $2\mathbf{1}_{\{|x|>1\}}(x)$. Using the fact that ν is a Lévy measure we can apply the dominated convergence theorem and prove the convergence.

Proof II

To prove the continuity at 0 we see that pointwise,

$$\lim_{t\downarrow 0}(e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x)) = 0$$

and fixing $t = 1$ the same bounds as before allows us to apply the dominated convergence theorem and prove that

$$\lim_{t\downarrow 0}\int_{\mathbb{R}}(e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x))\nu(dx) = 0$$

and so, φ is continuous at 0 and $\varphi(0) = 1$.

Proof III

Then, from the Lévy theorem of continuity,

$$\varphi(t) := \exp\{\int_{\mathbb{R}} (e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x))\nu(dx)\}$$

is the characteristic function of a probability measure, and we define it as the Compound Poisson measure associated to the Lévy measure ν and denote it by $P\nu$.

Remark: If ν is a measure not null at zero, but satisfies $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$, we can define $\bar{\nu} := \nu - \nu(\{0\})\delta_0$ that is a Lévy measure. And we can define a compound Poisson measure $P\nu$ because $P\nu = P\bar{\nu}$. Therefore, for any measure ν satisfying

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty$$

we can define an associated compound Poisson probability measure.

Lévy-Khinchin representation

The main result related with the family of infinite divisible laws is the celebrated Lévy-Khinchin representation that characterizes completely this family.

Before to establish the theorem we need some lemmas.

In all the section μ denotes an infinite divisible probability measure with characteristic function φ and characteristic exponent $\eta := \log \varphi$. The same for a sequence of infinite divisible probability measure μ_n.

Lévy-Khinchin theorem

Theorem: A probability measure μ is infinite divisible if and only if

$$\varphi(z) = \exp\{i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{\{|x|<1\}}(x))\nu(dx)\} \tag{2}$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a Lévy measure. Moreover, the triplet (γ, σ^2, ν) is unique.

The quantity

$$\eta(z) := i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{\{|x|<1\}}(x))\nu(dx)$$

is called the Lévy symbol or the Lévy characteristic of μ.

Remark

The function $g(x) = x\mathbf{1}_{\{|x|<1\}}(x)$ can be changed by any other function such that $e^{izx} - 1 - izg(x) \in L^1(\nu)$ for any $z \in \mathbb{R}$.

A typical alternative function is $g(x) = \frac{x}{1+x^2}$; see Itô (1956). The change of g implies a change of constant γ but σ^2 and ν are independent of the election of g.

Proof I

The proof can be found, for example, in Applebaum (2006) and Sato (1999).

For the direct implication assume we have a probability measure μ with the characteristic function φ given by the theorem. Note that φ is well defined thanks to ν is a Lévy measure. To show that μ is an infinite divisible probability law is immediate because φ is the characteristic function of the law.

$$N(\gamma, \sigma^2) * P\nu$$

and Normal and compound Poisson laws are infinite divisible and the convolution of infinite divisible laws is an infinite divisible law.

Proof II

To show the uniqueness of the canonical triple (γ, σ^2, ν) we proceed step by step proving the uniqueness of σ^2, then the uniqueness of measure ν and finally the uniqueness of γ.

For the reverse proof, we assume μ is infinite divisible and we construct a triplet (γ, σ^2, ν) and prove that the corresponding φ is given by the expression of the theorem

Moments I

The knowledge of the triplet of an infinite divisible distribution allows to obtain other properties. For example, moments can be characterized by the associated Lévy measure ν as the following proposition show.

Proposition: Let μ be an infinite divisible law and ν its associated Lévy measure. Let $u \in \mathbb{R}$. Therefore

- $$\int_{\mathbb{R}} |x|^n \mu(dx) < \infty \Leftrightarrow \int_{|x|\geq 1} |x|^n \nu(dx) < \infty$$
- $$\mathbb{E}(e^{uX}) < \infty \Leftrightarrow \int_{|x|\geq 1} e^{ux} \nu(dx) < \infty$$

Moments II

In particular, if X is a random variable with law μ we have

$$\mathbb{E}(X) = \gamma + \int_{|x|\geq 1} x\nu(dx)$$

and

$$\mathbb{V}(X) = \sigma^2 + \int_{\mathbb{R}} x^2\nu(dx).$$

In general, the moments can be computed differentiating the characteristic function.

Gaussian distributions

Let μ be an infinite divisible probability measure with triplet (γ, σ^2, ν).

Assume $\nu = 0$. In this case, the characteristic function of μ is given by

$$\varphi(z) = \exp\{i\gamma z - \frac{1}{2}\sigma^2 z^2\}$$

that is the characteristic function of a Gaussian law with mean γ and variance σ^2. This includes constant random variables as degenerate Gaussian ones.

Purely non Gaussian infinitely divisible distribution I

Assume $\sigma^2 = 0$. Then we have

$$\varphi(z) = \exp\{i\gamma z + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{\{|x|\leq 1\}}(x))\nu(dx)\}$$

and μ is called a purely non Gaussian infinite divisible law.

Purely non Gaussian infinitely divisible distribution II

Note that in the particular case of $\sigma^2 = 0$ and $\int_{|x|>1} |x|\nu(dx) < \infty$, we can write

$$\varphi(z) = \exp\{i\gamma z + \int_{\mathbb{R}} (e^{izx} - 1 - izx)\nu(dx) + iz \int_{|x|>1} x\nu(dx)\}$$

and if we call

$$\tilde{\gamma} := \gamma + \int_{|x|>1} x\nu(dx)$$

we can rewrite

$$\varphi(z) = \exp\{i\tilde{\gamma} z + \int_{\mathbb{R}} (e^{izx} - 1 - izx)\nu(dx)\}.$$

Thanks to a previous proposition, condition $\int_{|x|>1} |x|\nu(dx) < \infty$ is equivalent to $\int_{\mathbb{R}} |x|\mu(dx) < \infty$ and in this case

$$\tilde{\gamma} = \gamma + \int_{\mathbb{R}} x\mu(dx).$$

Compound Poisson distribution

On the other hand, if $\sigma^2 = 0$ and ν is finite, we can define $\lambda := \nu(\mathbb{R})$ and write $\nu = \lambda Q$ where Q is a probability measure. Thus, we have

$$\varphi(z) = \exp\{i\gamma z + \lambda \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{\{|x|\leq 1\}}(x))Q(dx)\}, \quad (3)$$

which is the characteristic function of a Compound Poisson law with intensity λ and jump law Q if

$$\gamma := \lambda \int_{|x|\leq 1} xQ(dx) = \int_{|x|\leq 1} x\nu(dx).$$

Note that if in the formula (3) we choose $Q := \delta_1$ we obtain the characteristic function of a Poisson random variable

$$\varphi(z) = \exp\{\lambda(e^{iz} - 1)\}.$$

Stochastic process

We assume we have a real-valued stochastic process $X = \{X_t, t \in \mathbb{T}\}$ defined on filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

The filtration $\mathbb{F}$ is assumed to be complete and right-continuous and $\mathcal{F} = \mathcal{F}_\infty$ if $\mathbb{T} = [0, \infty)$ and $\mathcal{F} = \mathcal{F}_T$ if $\mathbb{T} = [0, T]$.

We assume all processes have càdlàg or càglàd trajectories with probability one. This is not restrictive for our purposes.

Jump function

Given a càdlàg function f we can define the associated jump function

$$(\Delta f)(t) := f(t) - f(t-).$$

It is well-known that the set of jumps on an interval $[0, T]$, that we can write as $\{t \in [0, T] : (\Delta f)(t) \neq 0\}$, is countable.

Indeed, for any $\epsilon > 0$, the set $\{t \in [0, T] : |(\Delta f)(t)| \geq \epsilon\}$ is finite.

If we change $[0, T]$ by $[0, \infty)$, both sets are countable.

See for example Applebaum (2009).

Variation

Another important property of trajectories of a stochastic process is variation.

Given a function $f : [0, T] \longrightarrow \mathbb{R}$ and a partition π of $[0, T]$, we define the total variation of f with respect to π as

$$TV(f) := \sup_{\pi \in \Pi} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

where Π is the set of all partitions of $[0, T]$.

We say that f is a function of bounded or finite variation if $TV(f) < \infty$.We say that a process X is of bounded or finite variation if it has a.s. finite variation trajectories.

We say that f is a function of unbounded of infinite variation if $TV(f) = \infty$. We say that a process X is of unbounded or infinite variation if it has a.s. infinite variation trajectories.

Integrability

Finally, we introduce some notions related with integrability properties of an stochastic process.

Definition: We say that a process X is integrable of order $p \geq 1$ (or $p-$integrable) if $\mathbb{E}(|X_t|^p) < \infty$ for any $t \in \mathbb{T}$. In particular, for $p = 1$ we say the process is integrable. And for $p = 2$ we say the process is square-integrable.

Note that for any $p-$integrable process, $\mathbb{E}(X_t^p)$ is well defined for any $t \in \mathbb{T}$.

In particular, for a square-integrable process, the variance $\mathbb{V}(X_t) := \mathbb{E}(X_t^2) - (\mathbb{E}(X_t))^2$ is finite for any $t \in \mathbb{T}$ as a consequence of the Cauchy-Schwarz inequality.

Moreover, recall that we say that an integrable process is centered if $\mathbb{E}(X_t) = 0$ for any $t \in \mathbb{T}$.

Non-anticipation

Given a stochastic process X, a natural hypothesis in the classical Itô calculus is to assume that X is non-anticipating, that is, for any t, X_t a random variable whose value is known at t. Thus, the notion of filtration allows us to introduce the following definition:

Definition: We say that a process X is non-anticipating or adapted with respect the filtration $\mathbb{F} := \{\mathcal{F}_t, t \in \mathbb{T}\}$ if and only if X_t is $\mathcal{F}_t$-measurable for any $t \in \mathbb{T}$. In this case we say that X is $\mathbb{F}-$adapted.

Predictability, optionality and progressive measurability I

The following definition introduces the important notion of predictability.

Definition: Given a stochastic process X we have the following definitions:

a) X is called a predictable process if and only if it is measurable with respect the predictable $\sigma-$algebra $\mathcal{P}$, which is the $\sigma-$algebra on $\mathbb{T} \times \Omega$ generated by all adapted and càglàd processes

b) X is call an optional process if and only if it is measurable with respect to the optional $\sigma-$algebra $\mathcal{O}$, which is the $\sigma-$algebra on $\mathbb{T} \times \Omega$ generated by all adapted càdlàg processes.

c) X is a progressively measurable process if and only, for every $t \in \mathbb{T}$, the application

$$X : ([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) \to (\mathbb{R}, \mathcal{B})$$

is measurable.

Predictability, optionality and progressive measurability II

It is well-known that $\mathcal{P} \subset \mathcal{O}$. A predictable processes is adapted but the reverse is not true.

For example, the standard Poisson process N is a measurable and $\mathcal{F}_t^N$-adapted process but not predictable. Note that on a jump τ we have $N_\tau \neq N_{\tau-}$ and $N_\tau - N_{\tau-} = (\Delta N)_\tau = 1$.

Proposition: Let $X : [0, T] \times \Omega \to \mathbb{R}$ be an adapted and right-continuous, or left-continuous, process. Then, X is progressively measurable.

Note that this implies that predictable and optional process are also a progressively measurables.

On other hand any adapted process with continuous trajectories is also predictable and progressively measurable.

Gaussian processes

Definition: A process $X : \mathbb{T} \times \Omega \to \mathbb{R}$ is a Gaussian process if and only if their finite-dimensional probability measures are multidimensional Gaussian. That is, for any $n \geq 1$ and for any collection of increasing times $t_1, \ldots, t_n \in \mathbb{T}$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ has a multidimensional Normal law.

A very important example of Gaussian process is Brownian motion. A Brownian motion is a centered Gaussian process null at the origin and with covariance function $\mathbb{C}(X_s, X_t) = \sigma^2(s \wedge t)$ for a certain positive constant σ^2.

An extension of this notion, useful in the theory of additive processes, is the case the covariance function is $v(s \wedge t)$ where now v is an increasing function on $[0, \infty)$ with $v(0) = 0$.

The standard Poisson process I

A second important family is the family of Poisson processes. First of all we introduce the so-called standard Poisson process.

Let $\{T_i, i \geq 1\}$ a sequence of independent and identically distributed random variables with exponential distribution with parameter $\lambda > 0$. Define now $S_n := \sum_{i=1}^n T_i$. It is easy so see that S_n follows a Gamma distribution of parameters n and λ.

Definition: A standard Poisson process of intensity $\lambda > 0$ is a process $N = \{N_t, t \geq 0\}$ defined by

$$N_t := \sum_{n=1}^{\infty} \mathbf{1}_{[0,t]}(S_n) = \sum_{n=1}^{\infty} n \mathbf{1}_{[S_n, S_{n+1})}(t).$$

The standard Poisson process II

Note that N_t counts the number of elements of the sequence $\{S_n, n \geq 1\}$ before t. Note that we can write

$$S_n := \inf\{t \geq 0 : N_t = n\}.$$

Proposition: The process $\{N_t, t \geq 0\}$ satisfies the following properties:

- $\mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$.
- $\varphi_{N_t}(s) = e^{\lambda t (e^{is} - 1)}$
- $\mathbb{E}(N_t) = \mathbb{V}(N_t) = \lambda t$.

Compensated Poisson process

We can associate to the standard Poisson process N its compensated Poisson process

$$\tilde{N}_t := N_t - \lambda t.$$

Note that this new process is centered but has the same variance function than N, that is, λt. Compensated Poisson processes will be intensively used in this course.

Non-homogeneous Poisson process

An important enlargement of the notion of Poisson process is the so-called non-homogeneous Poisson process, that is, the non-stationary version of the previous one.

Indeed we can change the constant λ by an integrable function

$$\lambda : t \in [0, \infty) \longrightarrow \lambda(t) \in [0, \infty)$$

and assume $N_t - N_s$, for any $s \leq t$, is a Poisson random variable with parameter

$$\int_s^t \lambda(u) du$$

that coincides with the expectation and the variance of the process. Of course, in this case, the compensated process is

$$\tilde{N}_t = N_t - \int_0^t \lambda(s) ds.$$

Compound Poisson process

A natural generalization of the standard Poisson process is the so-called Compound Poisson process.

Consider now a sequence of independent and identically distributed random variables $\{Y_n, n \geq 1\}$ with associated probability measure P_Y. Let N be a standard Poisson process with intensity λ. A Compound Poisson process X is defined as

$$X_t := \sum_{i=1}^{N_t} Y_i$$

where the sum is 0 if $N_t = 0$. Note that the standard Poisson process is the particular case of $Y_i \equiv 1$, for any $i \geq 1$.

It is easy to see that $\mathbb{E}(X_t) = \lambda t \mathbb{E}(Y)$ and $\mathbb{V}(X_t) = \lambda t \mathbb{E}(Y^2)$.

Non homogeneous compound Poisson process

Of course we have the corresponding non-homogeneous Compound Poisson process, substituting the constant λ by an integrable function $\lambda : \mathbb{T} \to \mathbb{R}$. In this case, if

$$\Lambda(t) := \int_0^t \lambda(s)ds$$

we have

$$\mathbb{E}(\sum_{i=1}^{N_t} Y_i) = \Lambda(t)\mathbb{E}(Y)$$

and

$$\mathbb{V}(\sum_{i=1}^{N_t} Y_i) = \Lambda(t)\mathbb{E}(Y^2).$$

Interlacing

Let G be a Gaussian process and C be a Compound Poisson process, independent between them. Consider

$$X_t := G_t + C_t.$$

Note that if $T_1, \ldots, T_n$ are jump instants of C, $X_t = G_t$, for $t < T_1$, $X_t = G_t + Y_1$ if $t = T_1$, $X_t = X_{T_1} + G_t - G_{T_1}$ if $t \in (T_1, T_2)$, $X_{T_2} = G_{T_2} + Y_1 + Y_2$ and so on. In this process a continuous path is interlaced with random jumps.

As we will see, Lévy and additive processes are limits of sequences of such interlacings.

Martingales

Definition: A process X is an $\mathbb{F}$-martingale if and only if it satisfies the following three conditions:

1. X is an $\mathbb{F}$-adapted process.
2. X satisfies $\mathbb{E}(|X_t|) < \infty$, for any $t \in \mathbb{T}$, that is, X is an integrable process.
3. For $s, t \in \mathbb{T}$, $s \leq t$, $E(X_t|\mathcal{F}_s) = X_s$, a.s.

Note that the definition of a martingale depends on both the probability measure $\mathbb{P}$ and the filtration $\mathbb{F}$.

It can be proved that any martingale has a càdlàg version

The Brownian motion and the compensated Compound Poisson process introduced above are examples of martingales.

Local martingales

Finally, an extension of the notion of martingale that requires the notion of stopping time is given by the following definition

Definition: An adapted process $\bar{M} := \{\bar{M}(t), t \geq 0\}$ is a local martingale if there exists a sequence of stopping times $\{\tau_n, n \geq 1\}$, with $\tau_n \uparrow \infty$, such that for any n, the process $M_n := \{\bar{M}(t \wedge \tau_n), t \geq 0\}$ is a martingale.

Any martingale is a local martingale. Take for example the sequence $\tau_n := n$. But the reverse is false.

Semimartingales I

Another extension of the notion of martingale is given by the following definition.

Definition: We say that a process X is a semimartingale if and only if

$$X(t) = X(0) + \bar{M}(t) + C(t), t \geq 0$$

where $\bar{M}$ is a local martingale and C is an adapted process of finite variation, that is, a process with finite variation trajectories.

Semimartingales II

The importance of semimartingales, see Protter (2005) and Revuz-Yor (1991), is the fact that we can define stochastic integrals with respect to them. Given a semimartingale X, for any locally bounded function f we can define

$$\int_0^t f(s)dX(s) = \int_0^t f(s)d\bar{M}(s) + \int_0^t f(s)dC(s).$$

The first integral in the right hand side is well defined as an Itô integral and the second one is well defined as a Riemann-Stieltjes integral.

Lévy processes I

In this section we introduce the notion of Lévy process and comment their basic properties. Previously we recall briefly the necessary background.

Lévy processes are essentially processes with independent and stationary increments.

Today, Lévy processes is a well-know topic and several excellent books that treat it are available in the literature. I mention by historical order, Ikeda-Watanabe (1989), Bertoin (1996), Sato (1999), Kallenberg (2002), Jacod-Shiryaev (2003) and Applebaum (2004).

Lévy processes II

Definition: It is said that $X = \{X_t, t \geq 0\}$ is a Lévy process if it satisfies the following conditions:

i) It is null at the origin, that is $X_0 = 0$, a.s.
ii) It has *càdlàg* trajectories, that is the trajectories of X are right continuous and have left limits.
iii) It has independent increments, that is, for each $n \geq 0$ and each $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the random variables $X_{t_j} - X_{t_{j-1}}$, for $j = 1, \ldots, n$, are independent.
iv) It has stationary increments, or in other words, it is time homogeneous, that is, the distribution of $X_{s+t} - X_s$ is the same as the distribution of $X_t - X_0 = X_t$, for any $s \geq 0$.

Continuity in probability

The following proposition is important in relation with the extension to additive processes.

Proposition: A Lévy process X is continue in probability, that is, $\forall \epsilon > 0$ and $t \geq 0$

$$\lim_{s \to t} \mathbb{P}(|X(t) - X(s)| > \epsilon) = 0.$$

Proof: Note first of all that it is enough to prove $\lim_{s \downarrow 0} \mathbb{P}(|X(s)| > \epsilon) = 0$ because using the stationarity of the increments we have

$$\lim_{s \to t} \mathbb{P}(|X(t) - X(s)| > \epsilon) = \lim_{s \to t} \mathbb{P}(|X(t-s)| > \epsilon) = \lim_{u \to 0} \mathbb{P}(|X(u)| > \epsilon).$$

Hypothesis (ii) ensure that $\lim_{s \downarrow 0} |X(s)| = 0$ a.s. So, in particular, the limit is also true in probability. Then, for any ϵ,

$$\lim_{s \downarrow 0} \mathbb{P}(|X(s)| > \epsilon) = 0.$$

Lévy processes and infinite divisibility I

The first fundamental result that characterizes Lévy processes is the following theorem:

Theorem: Let X be a Lévy process. Then,

a) For any $t \geq 0$, X_t has an infinite divisible distribution.
b) In particular, if η is the Lévy symbol of X_1 and η_t is the Lévy symbol of X_t we have that $\eta_t(u) = t\eta(u)$.
c) If P is an infinite divisible probability measure with Lévy symbol η, it exists a unique Lévy process such that $P = \mathcal{L}(X_1)$.
d) If X is a Lévy process and $\mathbb{E}(|X_t|) < \infty$ for any $t \geq 0$, we have $\mathbb{E}(X_t) = t\mathbb{E}(X_1)$.

Canonical Levy triplet

As a conclusion, given a Lévy process, there exists a triplet (γ, σ^2, ν) with $\gamma \in \mathbb{R}$, $\sigma^2 > 0$ and ν a Lévy measure, such that

$$\mathbb{E}(e^{izX_t}) = \exp\{t\eta(z)\}$$

with

$$\eta(z) := i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx \mathbb{1}_{\{|x| \leq 1\}}(x))\nu(dx).$$

The triplet (γ, σ^2, ν), called the generating triplet, determines a unique Lévy process.

Corollary

If X is a Lévy process and $c \in \mathbb{R}$, then process Y defined as $Y_t = X_t + ct$ is also a Lévy process

Proof: Clearly

$$\varphi_{Y_t}(z) = \mathbb{E}(e^{iuct} e^{iuX_t}) = \mathbb{E}(e^{iuct})\mathbb{E}(e^{iuX_t}) = e^{t\eta_Y(u)}$$

with $\eta_Y(u) = icu + \eta_X(u)$ and this implies simply a change of γ by $\gamma + c$.

Lévy processes and martingales I

A very important question since the point of view of applications to Finance is to determine when a Lévy process, or a certain transformation of a Lévy process, is a martingale. The following proposition has important consequences in financial modeling.

Proposition: Let $X := \{X_t, t \geq 0\}$ be a Lévy process with characteristic triplet (γ, σ^2, ν). We have the following two statements:

- X is a martingale if and only if $\int_{|x|\geq 1} |x|\nu(dx) < \infty$ and
$$\gamma + \int_{|x|\geq 1} x\nu(dx) = 0.$$
- Process e^{X_t} is a martingale if and only if $\int_{|x|\geq 1} e^x \nu(dx) < \infty$ and
$$\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x\mathbf{1}_{\{|x|\leq 1\}})\nu(dx) = 0.$$

Jump process of a Lévy process

Given a Lévy process X we define its associated jump process as

$$\Delta X := \{\Delta X(t),\ t \geq 0\},$$

where $(\Delta X)(t) := X(t) - X(t-)$.

The following proposition shows that the jump process associated to a Lévy process is null almost surely at any fixed time t. The proof is an immediate consequence of the continuity in probability.

Proposition: If X is a Lévy process, for any $t \geq 0$ we have $(\Delta X)(t) = 0$ a.s.

Poisson random measure associated to a Lévy process

Now, we introduce the Poisson random measure associated to a Lévy process. Given a Lévy process X and $B \in \mathcal{B}_0$ we define

$$N(t, B) := c\{s \in [0, t] : (\Delta X)(s) \in B\}$$

where $c(A)$ denotes the cardinal of the set A.

Note that $N(t, B)$ is a positive random variable that counts the number of jumps in $[0, t]$ with values in B.

Properties of a Poisson random measures

In relation with $N(t, B)$, the following properties are straightforward:

a) For fixed t, $N(t, \cdot)$ is a random measure on $\mathbb{R}$. The $\sigma-$additivity comes from the $\sigma-$additivity of the counting measure.

b) For a fixed B, $N(\cdot, B)$ is an increasing pure jump process, null at the origin, that jumps a unit when X jumps with an amplitude in B.

c) We can associate to X the measures $\nu_t(B) := \mathbb{E}(N(t, B))$. The $\sigma-$additivity of ν_t comes from monotone convergence because random variables $N(t, B)$ are positive.

d) Given a Lévy process X, for any $B \in \mathcal{B}$, the associated process $N(\cdot, B)$ is a Lévy process. We have $\nu_t(B) := \mathbb{E}(N(t, B)) = t\mathbb{E}(N(1, B)) = t\nu_1(B)$. The Lévy measure $\nu(B) = \nu_1(B) = \mathbb{E}(N(1, B))$ is called the intensity of process X.

The Poisson integral I

Let X be a Lévy process. Up to now we have introduced the random variables $N(t,B)$ for any $t \geq 0$ and $B \in \mathcal{B}$ and studied its properties. Note that we can write

$$N(t,B) = \int_{\mathbb{R}} \mathbf{1}_B(x) N(t,dx) = \int_0^t \int_{\mathbb{R}} \mathbf{1}_B(x) N(ds,dx)$$

The goal now is to extend this definition to random variables like

$$N(t,f) = \int_{\mathbb{R}} f(x) N(t,dx) = \int_0^t \int_{\mathbb{R}} f(x) N(ds,dx)$$

for suitable functions f.

The Poisson integral II

For $f \in L^1(\nu)$ we is it possible to define the compensated Poisson integral as

$$\tilde{N}(t,f) := \int_{\mathbb{R}} f(x)\tilde{N}(t,dx) = \int_{\mathbb{R}} f(x)N(t,dx) - t\int_{\mathbb{R}} f(x)\nu(dx).$$

Both integrals in the right hand side are well defined as almost-sure limits of the same integrals on the domain $[-\varepsilon,\varepsilon]^c$.

As a particular case, if we fix $\varepsilon > 0$, we can denote by T_i^{ε} for $i \geq 1$ the jumps of X such that $(\Delta X)(T_i^{\varepsilon}) > \epsilon$. The number of these jumps is finite on any interval $[0,t]$. Then, for any real function f we can define

$$N_{\varepsilon}(t,f) := \int_{[-\varepsilon,\varepsilon]^c} f(x)N(t,dx) = \sum_{i=1}^{\infty} f((\Delta X)(T_i^{\varepsilon}))\mathbf{1}_{[0,t]}(T_i^{\varepsilon}).$$

The Poisson integral III

Moreover, if $f \in L^2(\nu)$ the integral

$$\tilde{N}(t,f) = \int_{\mathbb{R}} f(x)\tilde{N}(t,dx)$$

is well defined as an Itô integral thanks to the isometry

$$\mathbb{E}(\tilde{N}(t,f))^2 = t\int_{\mathbb{R}} f(x)^2\nu(dx).$$

If $f \in L^1(\nu) \cap L^2(\nu)$ both stochastic integrals coincide.

Previous results can be extended taking a predictable process u instead of a function f for processes in L^1 and L^2 on $\Omega \times \mathbb{R}$.

The Lévy-Itô decomposition VII

As a summary, we have the following result.

Theorem: Let X be a Lévy process with triplet (γ, σ^2, ν). The following decomposition can be established:

$$X_t = \gamma t + G_t + J_t$$

where G and J are two independent Lévy processes with triplets $(0, \sigma^2, 0)$ and $(0, 0, \nu)$ respectively.

Process G turns out to be a Brownian motion with covariance $\mathbb{E}(G_s G_t) = \sigma^2 (s \wedge t)$

The Lévy-Itô decomposition I

Process J is called a pure jump Lévy process and can be a.s. represented as

$$J_t = \int_0^t \int_{0<|x|\leq 1} x\tilde{N}(ds, dx) + \int_0^t \int_{|x|>1} xN(ds, dx)$$

where the convergence on the right hand side is a.s. uniform in t on every compact interval of $\mathbb{T}$.For simplicity, we will write from now on

$$\int_0^t \int_{|x|\leq 1} x\tilde{N}(ds, dx) := \lim_{\epsilon\downarrow 0} \int_0^t \int_{\epsilon<|x|\leq 1} x\tilde{N}(ds, dx)$$

The Lévy-Itô decomposition II

It is interesting to emphasize the following particular cases:

1. If $\int_{|x|>1} |x|\nu(dx) < \infty$ we can rewrite the decomposition as
$$J_t = \int_0^t \int_{|x|>1} x\nu(ds, dx) + \int_0^t \int_{\mathbb{R}} x\tilde{N}(ds, dx).$$
2. If $\int_{|x|\leq 1} |x|\nu(dx) < \infty$ we can rewrite
$$J_t = \int_0^t \int_{\mathbb{R}} xN(ds, dx) - \int_0^t \int_{|x|\leq 1} x\nu(dx)ds.$$
In this case, process $J_t = \int_0^t \int_{\mathbb{R}} xN(ds, dx)$ is a true pure jump process.
3. If $\nu(\mathbb{R}) < \infty$ the corresponding Lévy process is a Compound Poisson process where jump instants and jump heights are independent with intensity parameter $\nu(\mathbb{R})$ and law of jump heights $\nu(\mathbb{R})^{-1}\nu$.

Lévy processes as semimartingales

Finally, the following corollary guarantees that we can integrate functions with respect a Lévy process.
Corollary: A Lévy process is a semimartingale

Proof: Given a Lévy process X, from the Lévy-Itô decomposition we can write

$$X(t) = M(t) + C(t)$$

where

$$M(t) = G(t) + \int_0^t \int_{0<|x|\leq 1} x\tilde{N}(ds, dx)$$

is a martingale, and

$$C(t) = \gamma t + \int_0^t \int_{|x|>1} xN(ds, dx)$$

is a process of finite variation.

Additive processes

In this section we introduce the more general notion of additive process and comment their basic properties. Additive processes are essentially processes with independents increments. So, they are an extension of Lévy processes in the sense we allow non-stationary increments, or in other words, additive processes are time inhomogeneous Lévy processes.

Only a few book treat additive processes. As far as I know, Sato (1999) and Jacod-Shiryaev (2003). As it is already said, we want to develop a Malliavin-Skorohod type calculus for additive processes with financial applications in view.

Basic notions

It is said that X is an additive process if it satisfies the following conditions:

i) It is null at the origin, that is $X_0 = 0$, a.s.

ii) It has *càdlàg* trajectories, that is the trajectories of X are right continuous and have left limits.

iii) It has independent increments, that is, for each $n \geq 0$ and each $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the random variables $X_{t_j} - X_{t_{j-1}}$, for $j = 1, \ldots, n$, are independent.

iv) It is stochastically continuous, that is, for all $c > 0$ and for all $t \in \mathbb{T}$,

$$\lim_{s \to t} \mathbb{P}(|X(t) - X(s)| > c) = 0.$$

Basic results

In relation with Lévy processes, note that we have changed the homogeneity in time by the stochastic continuity. This condition implies that there are no fixed time discontinuities.

The first fundamental results that characterizes additive processes is the following theorem:

Theorem: If process X is an additive process, then, for any $t \geq 0$, $\mathcal{L}(X_t)$ is an infinite divisible probability measure. Conversely, if P is an infinite divisible probability measure it exists, unique in law, an additive process such that $P = \mathcal{L}(X_1)$.

Lévy-Itô decomposition

The second fundamental results is the following.

Theorem: Given an additive process, for every $t \geq 0$, there exists a triplet $(\gamma_t, \sigma_t^2, \nu_t)$, such that

$$\mathbb{E}(e^{izX_t}) = \exp\{i\gamma_t z - \frac{1}{2}\sigma_t^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{\{|x|\leq 1\}}(x))\nu_t(dx)\}.$$

So, the triplet $(\gamma_t, \sigma_t^2, \nu_t)$, called the generating triplet, determines uniquely an additive process in law.

Lévy triplet I

It can be proved, see Sato (1999) that

- $\{\gamma_t, t \in \mathbb{T}\}$ is a continuous function null at the origin,
- $\{\sigma_t^2, t \in \mathbb{T}$ is a continuous and increasing function null at the origin
- For any Borel set B such that $\bar{B} \cap \{0\} = \emptyset$, $\{\nu_t(B), t \in \mathbb{T}\}$ is a continuous and increasing function null at the origin. Recall that $\bar{B}$ denotes the minimal closed set including B.

Lévy triplet II

Given the family of Lévy measures $\{\nu_t, t \geq 0\}$, we can construct a measure ν on $[0, \infty) \times \mathbb{R}$ such that

$$\nu([0, t] \times B) := \nu_t(B),$$

for any $t \in \mathbb{T}$ and $B \in \mathcal{B}$.

Note that as a consequence of the continuity of $\nu(\mathbb{R})$ as a function of time, we have $\nu(\{t\} \times \mathbb{R}) = 0$, for any $t \geq 0$.

Note that a Lévy process coincides with the particular case such that $\gamma_t = \gamma_L t$, $\sigma_t^2 = \sigma_L^2 t$ and $\nu_t = t\nu_L$ with $\gamma_L \in \mathbb{R}$, $\sigma_L^2 > 0$ and ν_L a Lévy measure. In particular, $\nu = \ell \otimes \nu_L$ where ℓ denotes the Lebesgue measure on $[0, \infty)$.

The Lévy-Itô decomposition for additive processes

Theorem: Let X be an additive process with triplet (γ, σ^2, ν). The following decomposition can be established:

$$X_t = \gamma_t + G_t + J_t$$

where G and J are two independent Lévy processes with triplets $(0, v, 0)$ and $(0, 0, \nu)$ respectively.

Process G turns out to be a centered Gaussian process with covariance $\mathbb{E}(G_s G_t) = v(s \wedge t)$ with v and increasing function from $[0, \infty)$ to $[0, \infty)$, null at the origin

The Lévy-Itô decomposition for additive processes

Process J is called a pure jump additive process and can be a.s. represented as

$$J_t = \lim_{\epsilon \downarrow 0} \int_0^t \int_{\epsilon < |x| \le 1} x\tilde{N}(ds, dx) + \int_0^t \int_{|x|>1} xN(ds, dx)$$

where the convergence on the right hand side is a.s. uniform in t on every compact interval of $\mathbb{T}$.

For simplicity, we will write from now on

$$\int_0^t \int_{|x| \le 1} x\tilde{N}(ds, dx) := \lim_{\epsilon \downarrow 0} \int_0^t \int_{\epsilon < |x| \le 1} x\tilde{N}(ds, dx)$$

Particular cases of additive processes

1. If $\int_0^t \int_{|x|>1} |x|\nu(ds,dx) < \infty$, for any $t \geq 0$, we can rewrite the decomposition as

$$X_t = \gamma_t + \int_0^t \int_{|x|>1} x\nu(ds,dx) + G_t + \int_0^t \int_{\mathbb{R}} x\tilde{N}(ds,dx).$$

2. If $\int_0^t \int_{|x|\leq 1} |x|\nu(ds,dx) < \infty$, for any $t \geq 0$, we can rewrite the decomposition as

$$X_t = \gamma_t + G_t + \int_0^t \int_{\mathbb{R}} xN(ds,dx) - \int_0^t \int_{|x|\leq 1} x\nu(ds,dx).$$

In this case, process $J_t = \int_0^t \int_{\mathbb{R}} xN(ds,dx)$ is a true pure jump process.

Random measures I

Consider a topological measure space $(E, \mathcal{E}, \mu)$. Assume μ is diffuse and $\sigma-$finite. Consider $\mathcal{A}$ the family of Borel sets $A \in \mathcal{E}$ such that $\mu(A) < \infty$. Note that $\mathcal{A}$ includes the $\emptyset$, is closed by finite unions and differences, and any decreasing sequence of sets has its limit set in $\mathcal{A}$.

We will use the following three definitions:

Definition: A random measure M on $(E, \mathcal{E})$ is a measurable map

$$M : \Omega \times \mathcal{E} \longrightarrow [0, \infty]$$

such that

a) $\{M(\cdot, A), A \in \mathcal{A}\}$ is a set of random variables

b) $M(\omega, \cdot)$ is a.s. a positive $\sigma-$additive measure on $\mathcal{A}$.

Random measures II

Definition: We say that a random measure M is centered if, for any $A \in \mathcal{A}$, $\mathbb{E}(M(A)) = 0$.

Definition: We say that a centered random measure M is an independent square integrable measure if, for any $A_1, A_2 \in \mathcal{A}$,

$$\mathbb{E}(M(A_1)M(A_2)) = \mu(A_1 \cap A_2).$$

Note that this implies that if A_1 and A_2 are disjoint sets, $M(A_1)$ and $M(A_2)$ are independent random variables. Note also that in particular $\mu(A)$ is the variance of $M(A)$.

Gaussian random measure

Given a centered Gaussian process with independent increments $X = \{X_t, t \geq 0\}$, null at the origin, we can define an associated independent square integrable measure M_X on $\mathcal{B}([0,\infty))$ simply defining $M_X((s,t]) = X_t - X_s$. It is called Gaussian random measure.

In this case $\mu(s,t] = \mathbb{E}((X_t - X_s)^2)$.

We can define $v(t) := \mu(0,t] = \mathbb{E}(X_t^2)$ that of course is an increasing function null at the origin. It is straightforward to see $\mu(s,t] = v(t) - v(s)$.

Poisson random measure

Let X an additive process with triplet $(\gamma_t, \sigma_t^2, \nu_t)$. Let ν be the associated measure.

Given a set $A \in [0, \infty) \times \mathbb{R}_0$ we can introduce $N(A)$ as the number of jumps such that $(s, \Delta X_s(\omega)) \in A$. For any A with $\nu(A) < \infty$, the random variable $N(A)$ is standard Poisson with expectation $\nu(A)$, and if $A_1, \ldots, A_n$ are disjoint sets, the random variables $N(A_1), \ldots, N(A_n)$ are independent. Therefore, N can be considered a random measure on $[0, \infty) \times \mathbb{R}$ and it is called a Poisson random measure.

We can also introduce the random measure $\tilde{N}$ such that $\tilde{N}(A) := N(A) - \nu(A)$ provided $\nu(A) < \infty$. This is the so-called compensated Poisson random measure.

Integrability with respect the compensated Poisson random measure

In this time inhomogeneous case we have the same results as in the homogeneous case.

If u is a predictable process such that

$$\mathbb{E}\int_0^\infty \int_{\mathbb{R}} u(t,x)^2 \nu(dt,dx) < \infty,$$

the process

$$\int_0^t \int_{\mathbb{R}} u(t,x) \tilde{N}(ds,dx)$$

is a square integrable martingale and we have the isometry

$$\mathbb{E}(\int_0^t \int_{\mathbb{R}} u(t,x) \tilde{N}(ds,dx))^2 = \mathbb{E}\int_0^t \int_{\mathbb{R}} u(t,x)^2 \nu(dt,dx) < \infty.$$

Integrability with respect the compensated Poisson random measure

Moreover, if u satisfy

$$\mathbb{E}\int_0^\infty \int_{\mathbb{R}} |u(t,x)|\nu(dt,dx) < \infty,$$

the integral

$$\int_0^t \int_{\mathbb{R}} u(t,x)\tilde{N}(ds,dx)$$

is well defined almost surely.

If $u \in L^1(\nu) \cap L^2(\nu)$ both stochastic integrals coincide.