Lecture 4: Malliavin-Skorohod calculus for additive processes

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June 1, 2024

Abstract I

The first purpose of this chapter is to show that additive processes enjoy, in a generalized sense that will be precised, the so called chaotic representation property (CRP).

This fact opens the possibility to apply the machinery introduced in Lecture 2 to functionals of an additive process, and therefore, to construct a stochastic calculus of Malliavin-Skorohod type for additive processes.

For the Brownian motion case, the result was proved in Itô (1951), see also Nualart (2006) and Nualart and Nualart (2018).

For Lévy processes, the result appeared for the first time in Itô (1956). An alternative proof for pure jump Lévy processes was presented in Løkka (2004), see also Di Nunno-Øksendal-Proske (2009). For additive processes the first reference is Yablonski (2008).

Abstract II

Moreover, we introduce in this lecture a canonical space for additive processes. On this canonical space, the operators introduced in Lecture 2 will have a nice probabilistic interpretation in terms of derivatives, differences or integrals.

For the Gaussian part, the canonical space will be the well-known canonical space of the Brownian motion.

For the pure jump part, the canonical space introduced here is more convenient to our purposes than the more traditional canonical space of càdlàg trajectories. It is inspired on the canonical space for the standard Poisson process introduced by Neveu (1977). This canonical space, under the context of Lévy processes can be found in Solé-Utzet-Vives (2007), Di Nunno-Vives (2016) and Solé-Utzet (2016).

Finally, we want to introduce a probabilistic interpretation of the annihilation and creation operators D and δ introduced in Lecture 2 in the framework of additive processes.

Lévy triplet I

Consider an additive process $X = \{X_t,\ t \geq 0\}$ with triplet $(\gamma_t, \sigma_t^2, \nu_t),\ t \geq 0$. We denote $\Theta := [0,\infty) \times \mathbb{R}$, $\mathbb{R}_0 := \mathbb{R} - \{0\}$ and $\Theta_{\infty,0} := [0,\infty) \times \mathbb{R}_0$. Clearly we have

$$\Theta = ([0,\infty) \times \{0\}) \cup \Theta_{\infty,0}$$

endowed with the Borel $\sigma-$algebra. Moreover

$$[0,\infty) \times \{0\} \simeq [0,\infty).$$

Lévy triplet II

Recall that we have two measures associated to the additive process X.

On one hand, the measure ν on Θ, such that, for any Borel set $B \in \mathcal{B}(\mathbb{R})$, we have $\nu([0,t] \times B) := \nu_t(B)$. Being $\nu_t(\{0\}) = 0$ for any t, when convenient, we can consider ν as a measure restricted on $\Theta_{\infty,0}$. Hypotheses on ν_t guarantee that $\nu(\{t\} \times B) = 0$, for any $t \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$, and in particular, the fact that ν is nonatomic and $\sigma-$finite.

On other hand, we have the measure β on $[0,\infty)$ such that $\beta((s,t]) = \sigma_t^2 - \sigma_s^2$, for any interval $(s,t]$ with $s \leq t$. Recall that function σ^2 is continuous, null at the origin and non decreasing and this guarantees also that β is nonatomic and σ-finite.

Associated measures I

Now we can consider on Θ the nonatomic and $\sigma-$finite Borel measure

$$\mu(dt, dx) := \beta(dt)\delta_0(dx) + \nu(dt, dx)$$

defined on $\mathcal{B}(\Theta)$.

So, for $E \in \mathcal{B}(\Theta)$,

$$\mu(E) = \int_{E(0)} \beta(dt) + \iint_{E'} \nu(dt, dx),$$

where $E(0) = \{t \geq 0 : (t, 0) \in E\}$ and $E' = E - E(0)$.

Note that μ is also continuous in the sense that $\mu(\{t\} \times B) = 0$, for all $t \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$.

Associated measures II

Then, for $E \in \mathcal{B}(\Theta)$ with $\mu(E) < \infty$, we can define the following random measure

$$M(E) = \int_{E(0)} dW_t + L^2 - \lim_{n\uparrow\infty} \iint_{\{(t,x)\in E: \frac{1}{n}<|x|<n\}} \widetilde{N}(dt, dx),$$

that is, a centered and independent random measure such that

$$\mathbb{E}\left[M(E_1)M(E_2)\right] = \mu(E_1 \cap E_2),$$

for $E_1, E_2 \in \mathcal{B}(\Theta)$ with $\mu(E_1) < \infty$ and $\mu(E_2) < \infty$.

The measure M is a mixture of a Gaussian random measure and a compensated Poisson random measure. We can write

$$M(dt, dx) = (W \otimes \delta_0)(dt, dx) + \tilde{N}(dt, dx).$$

Remark

A similar framework can be developed with measures

$$\bar{\mu}(dt, dx) = \beta(dt)\delta_0(dx) + x^2\nu(dt, dx)$$

and

$$\bar{M} = (W \otimes \delta_0) + x\tilde{N}$$

as can be seen in Itô (1956) and Solé-Utzet-Vives (2007), for Lévy processes.

Multiple stochastic integrals I

Given measure μ, we can consider, for $n \geq 1$, the Hilbert spaces

$$L^2(\Theta^n) := L^2\left(\Theta^n, \mathcal{B}(\Theta)^{\otimes n}, \mu^{\otimes n}\right)$$

with the usual scalar product

$$\langle f, g\rangle_n := \int_{\Theta^n} fg d\mu^{\otimes n}.$$

By convention, we consider $L^2(\Theta^0) := \mathbb{R}$. Recall the well-known isometry $L^2(\Theta^n) \approx (L^2(\Theta))^{\otimes n}$.

Define $I_0(c) = c$, for any $c \in \mathbb{R}$.

Multiple stochastic integrals II

For $m \geq 1$, we define the Itô multiple stochastic integrals $I_m(f)$ with respect to measure M, for functions f in $L^2(\Theta^m)$.

Consider $f := \mathbf{1}_{E_1 \times \cdots \times E_m}$ with $E_1, \ldots, E_m \in \mathcal{B}(\Theta)$ pairwise disjoint and with finite measure μ. Define $I_m(f) := M(E_1) \cdots M(E_m)$. By linearity it is immediate to extend the previous definition to the set $\mathcal{E}_m$ of elementary functions of type

$$f = \sum_{i_1, \ldots, i_m = 1}^{n} c_{i_1, \ldots, i_m} \mathbf{1}_{E_{i_1} \times \cdots \times E_{i_m}}$$

where we assume that constants $c_{i_1, \ldots, i_m}$ are null if two of the indexes are equal.

Note that functions of $\mathcal{E}_m$ are null at the diagonals.

Multiple stochastic integrals III

It can be shown that for $f \in \mathcal{E}_m$ we have $\mathbb{E}(I_m(f)^2) = m!||\tilde{f}||_m^2 \leq m!||f||_m^2$, and this inequality allows to extend I_m from $\mathcal{E}_m$ to the whole $L^2(\Theta^m)$.

These integrals have the following properties:

- For any $f \in L^2(\Theta^n)$, we have $I_n(f) = I_n(\tilde{f})$, where $\tilde{f}$ is the symmetrization of f.
- $$\mathbb{E}[I_n(f)I_m(g)] = \mathbf{1}_{\{m=n\}} n! \int_{\Theta^n} \tilde{f}\tilde{g} d\mu^{\otimes n},$$

  for any $f \in L^2(\Theta^n)$ and $g \in L^2(\Theta^m)$.

  Note that, in particular, this shows that $\mathbb{E}(I_n(f)) = 0$, for any $f \in L^2(\Theta^n)$ and $n \geq 1$.

Multiple stochastic integrals IV

The properties of multiple stochastic integrals allow to built the following isometries.

Consider, for any $n \geq 1$, the space $L^2_s(\Theta^n)$ as the subspace of $L^2(\Theta^n)$ of symmetric functions. This subspace is a Hilbert space endowed with the scalar product

$$\langle \tilde{f}, \tilde{g} \rangle_{L^2_s(\Theta^n)} = n! \langle \tilde{f}, \tilde{g} \rangle_{L^2(\Theta^n)} \tag{1}$$

and we have the isometry

$$L^2_s(\Theta^n) \approx (L^2(\Theta))^{\odot n}$$

where here $\odot$ denotes the symmetric tensor product endowed with the scalar product (1).

Multiple stochastic integrals V

Define $C_0 := \mathbb{R}$ and $C_n := \{I_n(f) : f \in L_s^2(\Theta^n)\}$. This is a Hilbert subspace of $L^2(\Omega)$ endowed by the scalar product

$$E(I_n(f)I_n(g)) = n!\langle f, g, \rangle_{L^2(\Theta^n)} = \langle f, g, \rangle_{L_s^2(\Theta^n)}.$$

So, I_n establishes an isometry between $L_s^2(\Theta^n)$ and C_n, for any n.

Chaotic representation property I

Finally we can conclude that process X has the so-called chaos representation property.

Theorem: For any functional $F \in L^2(\Omega, \mathcal{F}^X, \mathbb{P})$, where $\mathcal{F}^X$ is the completed natural filtration of X, we have

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

for a certain unique family of symmetric kernels $f_n \in L^2_s(\Theta^n)$. Note that $f_0 := \mathbb{E}(F)$

Chaotic representation property II

Previous decomposition is called usually the chaos expansion of functional F. We will always assume that kernels of such decomposition are symmetric.

Thanks to the linearity and orthogonality of multiple stochastic integrals, previous Theorem shows that the space of square integrable functionals can be decomposed in a sum of orthogonal spaces in the following way

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty} C_n \approx \bigoplus_{n=0}^{\infty} L_s^2(\Theta^n) \approx \bigoplus_{n=0}^{\infty} L^2(\Theta)^{\odot n}.$$

This allows to see $L^2(\Omega)$ as a Fock space and so we can apply to square integrable functionals of an additive process all the machinery related with annihilation and creation operators in a Fock space as introduced in Lecture 2.

Chaotic representation property III

Proof: It is clear that $\bigoplus_{n=0}^{\infty} C_n \subseteq L^2(\Omega)$.
The difficult point is to show the completeness of the decomposition, that essentially means that we have to show that if $\xi \in L^2(\Omega)$ and $\mathbb{E}(\xi I_n(f_n)) = 0$ for any n and f_n this implies $\xi = 0$.
For Lévy processes, in Itô (1956), (see also Nualart and Nualart (2018) or Di Nunno-Øksendal-Proske), it is shown that simple multiple stochastic integrals, that are products of type

$$\prod_{i=1}^{m}(N(E_i) - \nu(E_i)) \cdot \prod_{j=1}^{n} \Delta W(I_j)$$

with E_i and I_j all disjoint subsets of Θ with finite measure μ, are a total set in $L^2(\Omega)$.
Remember that total means that linear combinations of elements of the class are dense in $L^2(\Omega)$. The extension to additive processes is quite straightforward.

Chaotic representation property IV

An alternative method is to identify C_n with the set $\mathcal{P}_n$ of polynomials of order n of a certain family and show that theses polynomials are a total family of $L^2(\Omega)$.

For the Gaussian case this is done by Itô (1951) using Hermite polynomials. For the Poisson case this was done by Ogawa (1971) using Charlier polynomials.

For the additive case this is done by Yablonski (2008) using a suitable general family of orthogonal polynomials built specifically in the paper for the so-called isonormal additive processes. Lévy processes are particular cases of isonormal additive processes and Hermite and Charlier polynomials are particular cases of the constructed family of orthogonal polynomials. Strictly speaking, this last paper is the only proof really suited to additive processes I know.

A canonical pure jump additive process

Consider in this subsection an additive process $J := \{J_t, t \geq 0\}$ with triplet $(0, 0, \nu_t)$. As we know, we can write

$$J_t = \int_0^t \int_{|x|>1} xN(ds, dx) + \int_0^t \int_{|x|\leq 1} x\tilde{N}(ds, dx).$$

Recall that given the family of Lévy measures $\{\nu_t, t \geq 0\}$ we can consider an associated measure ν on $[0, \infty) \times \mathbb{R}$ such that, for any Borel set $B \in \mathcal{B}(\mathbb{R})$, $\nu([0, t] \times B) := \nu_t(B)$. In all this subsection ν denotes this measure.

We want to construct a canonical space useful for our purposes. We follow Di Nunno and Vives (2017).

Finite activity in a compact interval I

Assume ν is concentrated in $\Theta_{T,\epsilon} = [0, T] \times \{|x| > \epsilon\}$ or otherwise consider the restriction of ν in this subset of Θ. Clearly, $\nu(\Theta_{T,\epsilon}) < \infty$. In this case,

$$J_t = \int_0^t \int_{\mathbb{R}} xN(ds, dx) - \int_0^t \int_{|x|\leq 1} x\nu(ds, dx)$$

We can define

$$c_t := \int_0^t \int_{|x|\leq 1} x\nu(ds, dx)$$

and this integral is bounded by $\nu_t([-1, 1])$.

Then, if we define $Y_t := J_t + c_t$, it is clear that

$$Y_t := \int_0^t \int_{\mathbb{R}} xN(ds, dx)$$

Finite activity in a compact interval II

It is clear that process Y is a time inhomogeneous compound Poisson process with intensity $\nu(\Theta_{t,\epsilon})$ and has a finite number of jumps in $[0, T]$.

Any trajectory of this process is totally described by a collection of pairs (t_j, x_j) where times t_j denote jump instants and values x_j denote the corresponding jump highs.

The space of this collection of pairs will be the canonical space of Y and as a consequence the canonical space of J, being the difference only the deterministic function c_t.

Finite activity in a compact interval III

For this process we can define the following probability space:

a) A set Ω of finite sequences of pairs (t_i, x_i), that is,

$$\Omega^J_{T,\epsilon} := \cup_{n=0}^{\infty} \Theta^n_{T,\epsilon},$$

where, for any $n \geq 0$, $\Theta^n_{T,\epsilon}$ is the set of all possible trajectories ω with n jumps greater than ϵ. In particular $\Theta^0_{T,\epsilon}$ denotes the set of the unique no-jump trajectory that we can denote by α.

b) The $\sigma-$algebra $\mathcal{F}_{T,\epsilon} := \{B \subseteq \Omega^J_{T,\epsilon} : B = \cup_{n=0}^{\infty} B_n, B_n \in \mathcal{B}((\Theta_{T,\epsilon})^n)\}$.

c) We define the probability measure $\mathbb{P}_{T,\epsilon}$ such that, for a given $B = \cup_{n=0}^{\infty} B_n$ of $\mathcal{F}_{T,\epsilon}$, we have

$$\mathbb{P}_{T,\epsilon}(B) = e^{-\nu(\Theta_{T,\epsilon})} \sum_{n=0}^{\infty} \frac{\nu^{\otimes n}(B_n)}{n!}, \quad \nu^{\otimes 0} := \delta_\alpha.$$

Finite activity in a compact interval IV

Given this probability space, we define, for any $\omega := ((t_1, x_1), \dots, (t_n, x_n)) \in \Omega^J_{T,\epsilon}$, the canonical process

$$\bar{Y}_t := \sum_{j=1}^{n} x_j \mathbf{1}_{[0,t]}(t_j), \quad \bar{Y}_t(\alpha) = 0.$$

Finite activity in a compact interval V

Given a measurable space $(E, \mathcal{E})$, the collection of sets

$$\mathcal{E}^n_{sym} := \{C \in \mathcal{E}^{\otimes n} : C \text{ is symmetric } \}$$

is a $\sigma-$algebra, and a function $f : E^n \longrightarrow \mathbb{R}$ is $\mathcal{E}^{\otimes n}_{sym}-$measurable if and only if it is $\mathcal{E}^{\otimes n}-$measurable and symmetric.

Then, we can introduce on $\Omega_{T,\epsilon}$ the $\sigma-$algebra

$$\mathcal{F}_{T,\epsilon,sym} := \{A \subseteq \Omega : A = \cup_{n=0}^{\infty} A_n, A_n \in \mathcal{B}(\Theta_{T,\epsilon})^n_{sym}\}.$$

It is easy to see that $\mathcal{F}^J_{T,\epsilon} = \mathcal{F}_{T,\epsilon,sym}$.

Extension of the canonical space to $\Theta_{\infty,0}$ I

Now, we extend the previous canonical space to the case J is defined on $\Theta_{\infty,0} := [0,\infty) \times \mathbb{R}_0$.

It is easy to see that $\Omega^J_{T,\epsilon}$ is a metric space. Then we can define the sequence of spaces

$$(\Omega^J_m, \mathcal{F}^J_m, \mathbb{P}_m)$$

such that $T = m$ and $\epsilon = \frac{1}{m}$.

Note that the sequences of sets Θ_m and Ω^J_m are increasing. Moreover, $\Theta_{\infty,0} = \bigcup_{m\geq 1} \Theta_m$ is an increasing union of sets and

$$\Theta_{\infty,0} = \bigcup_{m\geq 1} (\Theta_m - \Theta_{m-1})$$

is a union of pairwise disjoint sets.

Extension of the canonical space to $\Theta_{\infty,0}$ II

The canonical space Ω^J can be constructed as the unique projective limit of the sequence of spaces $\{\Omega^J_m, m \geq 1\}$, see Parthasaraty (1965). We skip the details, that are very technical.

As a summary, $\Omega^J = \cup_{n=0}^{\infty} \Theta^n_{\infty,0}$ and the probability measure $\mathbb{P}$ is concentrated on the subset of

- The empty sequence α, corresponding to the element $(\alpha, \alpha, \dots)$.
- All finite sequences of pairs (t_i, x_i).
- All infinite sequences of pairs (t_i, x_i) such that for every T and ϵ, there is only a finite number of (t_i, x_i) on any $\Theta_{T,\epsilon}$.

The associated $\sigma-$algebra is $\mathcal{F}_{sym} := \{A \subseteq \Omega : A = \cup_{n=0}^{\infty} A_n, A_n \in \mathcal{B}(\Theta^n_{\infty,0})_{sym}\}$.

Canonical Gaussian space

Given a variance process $\sigma^2 := \{\sigma_t^2, t \geq 0\}$, that is, a continuous, null at the origin and increasing function, we consider the following canonical space for an additive process with triplet $(0, \sigma_t^2, 0)$.

The space $\Omega^W = C_0([0,\infty))$ is the space of continuous functions, null at the origin, with the topology of the uniform convergence on the compacts. The $\sigma-$algebra $\mathcal{F}^W$ is the corresponding Borel $\sigma-$algebra and the probability measure $\mathbb{P}_\sigma^W$ is the probability measure that makes the set of projections

$$\bar{W}_t^\sigma : \Omega^W \longrightarrow \mathbb{R}$$

be a centered Gaussian process with independent increments and variance process σ^2.

Additive canonical space

Take a triplet $(\gamma_t, \sigma_t^2, \nu_t)$. Consider $(\Omega^W, \mathcal{F}^W, \mathbb{P}_\sigma^W)$ as the canonical Wiener space associated to process σ^2 introduced in the previous section. Consider $(\Omega^J, \mathcal{F}^J, \mathbb{P}_\nu^J)$ the corresponding canonical space associated to Lévy measures ν_t. Let

$$\bar{J}^\nu : \Omega^J \longrightarrow \mathbb{R}$$

the canonical additive pure-jump process associated to ν.

Then we define the canonical space for an additive process as the product space

$$(\Omega^W \times \Omega^J, \mathcal{F}^W \otimes \mathcal{F}^J, \mathbb{P}_\sigma^W \otimes \mathbb{P}_\nu^J).$$

Additive canonical space

If we define

$$W_t(\omega^W, \omega^J) := \bar{W}_t^{\sigma}(\omega^W)$$

and

$$J_t(\omega^W, \omega^J) := \bar{J}_t^{\nu}(\omega^J),$$

the canonical additive process is

$$X_t := \gamma_t + W_t + J_t, \; t \geq 0.$$

Beyond the canonical space I

Consider price evolutions. If we have a unique price we can identify market scenarios as price evolutions.

But if we have different prices, we can also identify market scenarios with multidimensional trajectories but it is more rich to consider a set Ω of market scenarios and different price evolutions as different processes defined on Ω.

Note that when we work with a unique stochastic process (the same for random variables) we always can identify Ω with the canonical space without loosing generality.

Beyond the canonical space II

In our case, we observe trajectories of an additive process. Price evolutions for example. So, our Ω is the space of trajectories of an additive process. That is a subspace of càdlàg functions from $[0, \infty)$ to $\mathbb{R}$.

We have seen that these trajectories can be decomposed in a sum of three independent components, a deterministic function γ, a continuous trajectory null at the origin and a pure stepwise trajectory with certain properties, that can be identified with a sequence of pairs of jump times and jump sizes.

Despite to use the canonical space is useful to understand deeply the action of Malliavin-Skorohod operators, the scope of Malliavin-Skorohod calculus is wider and we can avoid the dependence of the canonical space. See Steinicke (2014). As we have commented above, this is necessary if we want to consider different additive processes.

Malliavin-Skorohod operators for multiple stochastic integrals

Let us introduce, first of all, the operators in the setting of multiple stochastic integrals.

Consider an additive process $X = \{X_t, t \geq 0\}$ with triplet $(\gamma_t, \sigma_t^2, \nu_t)$.

We assume $t \in [0, \infty)$, but of course, things can be re-written with $t \in [0, T]$ for $T > 0$.

For any $F \in L^2(\Omega, \mathcal{F}^X, \mathbb{P})$, we have

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

for a certain unique family of symmetric kernels $f_n \in L_s^2(\Theta^n)$.

Recall that I_n denotes the multiple stochastic integral of f_n with respect the random measure M with intensity μ introduced in previous lectures, and $\Theta = [0, \infty) \times \mathbb{R}$.

The gradient operator I

From the isometry between f_n and $I_n(f_n)$ we can introduce the following gradient operator, that is the version of the abstract annihilation operator introduced in Lecture 2 in the new context of functionals of an additive processes.

Take $F \in L^2(\Omega)$. Let $\{f_n, n \geq 0\}$ be its associated symmetric kernels. Assume

$$\sum_{n=1}^{\infty} n n! ||f_n||^2_{L^2(\Theta^n)} < \infty. \tag{2}$$

We say that functionals that satisfy this condition belong to *DomD*.

The gradient operator II

For any $F \in DomD$ we can define the gradient operator

$$D : F \in L^2(\Omega) \longrightarrow DF \in L^2(\Theta \times \Omega)$$

such that

$$D_\theta F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, \theta)), \ \theta \in \Theta$$

where the convergence of the series is in $L^2(\Theta \times \Omega, \mu \otimes \mathbb{P})$.

The gradient operator III

Notice that the condition to belong to $DomD$ guarantees that $D_\theta F$ is well-defined and that operator D is a closed and unbounded operator as we have seen before.

Note also that $DomD$ is a Hilbert space with the scalar product

$$\langle F, G\rangle := \mathbb{E}(FG) + \mathbb{E}\int_\Theta D_\theta F D_\theta G \mu(d\theta).$$

The divergence operator I

The corresponding creation operator δ acts now in the following way.

Let $u \in L^2(\Omega \times \Theta)$. We can represent process u as

$$u(\theta) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, \theta))$$

with $f_n \in L^2(\Theta^{n+1})$ and symmetric with the first n variables. Denote by $\tilde{f}_n$ the symmetrization of f_n.

The divergence operator II

Define $Dom\delta$ as the subspace of processes of $L^2(\Omega \times \Theta)$ such that

$$\sum_{n=0}^{\infty}(n+1)!||\tilde{f}_n||^2_{L^2_s(\Theta^{n+1})} < \infty. \quad (3)$$

For processes of $Dom\delta$ we can define

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

that clearly is a well-defined sum in $L^2(\Omega)$.

The duality formula

As we have proved in Lecture 2, for any $u \in Dom\delta$ and $F \in DomD$ we have the duality formula

$$\mathbb{E}[F\delta(u)] = \mathbb{E}\int_{\Theta} u(\theta) D_\theta F \mu(d\theta).$$

Other formulas seen in Lecture 2, converted to the language of multiple stochastic integrals are

1. Given $F \in L^2(\Omega)$ we have

$$\mathbb{E}(F|\mathcal{F}_t) = \sum_{n=0}^{\infty} I_n(f_n(\theta_1, \ldots, \theta_n)\mathbf{1}_{\{\max_{1\leq i\leq n} t_i < t\}}).$$

2. For any $s, t \geq 0$, we have

$$D_{t,x}\mathbb{E}(F|\mathcal{F}_s) = \mathbb{E}(D_{t,x}F|\mathcal{F}_s)\mathbf{1}_{[0,s]}(t).$$

   Note that this implies that if F is $\mathcal{F}_s-$measurable and $t > s$, $D_{t,x}F = 0$.

Malliavin derivative for multiple stochastic integrals I

So, we have now definitions of operators D and δ on the spaces $L^2(\Omega)$ and $L^2(\Omega \times \Theta)$ and we want to give a probabilistic interpretation of them.

Recall from previous lectures that in the canonical space we have the decomposition $\Omega = \Omega^W \times \Omega^J$ and our functionals can be written as $F(\omega^W, \omega^J)$ where ω^W represents a random selection of a continuous trajectory under the law $\mathbb{P}^W$ and ω^J is the random selection of a path-wise trajectory under the law $\mathbb{P}^J$.

So, in particular, we can see elements of $L^2(\Omega^W \times \Omega^J)$ as elements of $L^2(\Omega^W, L^2(\Omega^J))$, or alternatively, of $L^2(\Omega^J, L^2(\Omega^W)$. In both cases we have random elements taking values in a Hilbert space.

Malliavin derivative for multiple stochastic integrals II

Note that given $f : \Theta^n \to \mathbb{R}$, positive or $\mu^{\otimes n}$ integrable, and denoting $\theta_n = (s, x)$, we have

$$\begin{aligned}\int_\Theta f d\mu^{\otimes n} &= \int_{[0,\infty)\times\{0\}} \int_{\Theta^{n-1}} f(\theta_1, \dots, \theta_{n-1}, (s, 0)) d\mu^{\otimes n-1}(\theta_1, \dots, \theta_{n-1}) \beta(ds) \delta_0(dx) \\ &+ \int_{[0,\infty)\times\mathbb{R}_0} \int_{\Theta^{n-1}} f(\theta_1, \dots, \theta_{n-1}, (s, x)) d\mu^{\otimes n-1}(\theta_1, \dots, \theta_{n-1}) \nu(ds, dx).\end{aligned}$$

The Gaussian particular case

Assume function β associated to X is not null. We can consider the space $DomD^W$ as the set of functionals $F \in L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} nn! \int_{[0,\infty)\times\{0\}} \int_{\Theta^{n-1}} f(\theta_1, \dots, \theta_{n-1}, (s,0))^2 \beta(dt)\delta_0(dx) d\mu^{\otimes n-1}(\theta_1, \dots, \theta_{n-1}) < \infty$$

and, for $F \in DomD^W$, to define

$$D_{t,0}F := \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, (t,0)))$$

as an element of $L^2([0,\infty) \times \Omega, \beta \otimes \mathbb{P})$.

The pure jump particular case

Analogously, if the associated measure ν is not null, we can define the set $DomD^J$ of functionals of $L^2(\Omega^W \times \Omega^J)$ such that

$$\sum_{n=1}^{\infty} nn! \int_{[0,\infty)\times\mathbb{R}_0} \int_{\Theta^{n-1}} f(\theta_1, \ldots, \theta_{n-1}, (s,x))^2 d\mu^{\otimes n-1} \nu(ds, dx) < \infty$$

and in this case to define

$$D_\theta F := \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, \theta))$$

as an element of $L^2([0,\infty) \times \mathbb{R}_0 \times \Omega, \nu \otimes \mathbb{P})$.

The general case

It is immediate to see that in the case of measures β and ν are not null we have

$$DomD = DomD^W \cap DomD^J.$$

Note that in the general case, measures β and ν are not null. In this case, the operator D, for a $\theta \in \mathbb{R}$, is $D_{t,0}$ or $D_{t,x}$, for $x \in \mathbb{R}_0$.

So, to give a probabilistic interpretation of D is equivalent to give probabilistic interpretations of operators $D_{t,0}$ and $D_{t,x}$ for a $x \in \mathbb{R}_0$.

Note that if measure β is null we have a pure jump additive process and $D_{t,0}F = 0$ and if measure ν is null, the additive process becomes a Gaussian process with drift and $D_{t,x}F = 0$, for any $x \neq 0$.

The Clark-Haussmann-Ocone formula I

Let $A \in \mathcal{B}(\Theta)$ and $\mathcal{F}_A := \sigma\{M(A') : A' \in \mathcal{B}(\Theta), A' \subseteq A\}$.

- F is $\mathcal{F}_A-$measurable if for any $n \geq 1$, $f_n(\theta_1, \ldots, \theta_n) = 0$, $\mu^{\otimes n} - a.e.$ unless $\theta_i \in A \quad \forall\, i = 1, \ldots n$.
- In particular, we are interested in the case $A = \Theta_{t-} := [0, t) \times \mathbb{R}$. Denote $\mathcal{F}_{t-} := \mathcal{F}_{\Theta_{t-}}$. Obviously, if $F \in \mathrm{Dom}\, D$ and it is $\mathcal{F}_{t-}-$measurable then $D_{s,x}F = 0$ for a.e. $s \geq t$ and any $x \in \mathbb{R}$.

The Clark-Haussmann-Ocone formula II

From the chaos representation property we can see that for $F \in L^2(\Omega)$,

$$E[F|\mathcal{F}_{t-}] = \sum_{n=0}^{\infty} I_n(f_n(\theta_1, \dots, \theta_n) \prod_{i=1}^{n} \mathbf{1}_{[0,t)}(t_i)).$$

Then, for $F \in DomD$ we have

$$D_{s,x} E[F|\mathcal{F}_{t-}] = E[D_{s,x} F|\mathcal{F}_{t-}] \mathbf{1}_{[0,t)}(s), \ (s,x) \in \Theta.$$

The Clark-Haussmann-Ocone formula III

Using these facts we can prove the very well known CHO formula:

If $F \in DomD$ we have

$$F = \mathbb{E}(F) + \delta(E[D_{t,x}F|\mathcal{F}_{t-}]).$$

- Note that being the integrand a predictable process, the Skorohod integral δ here above is actually an Itô integral.
- Note also that the CHO formula can be rewritten in a decompactified form as

$$F = \mathbb{E}(F) + \int_0^\infty E(D_{s,0}F|\mathcal{F}_{s-})dW_s + \int_{\Theta_{\infty,0}} E(D_{s,x}F|\mathcal{F}_{s-})\tilde{N}(ds, dx).$$

The Gaussian case I

Assume now that $W = \{W_t, t \geq 0\}$ is an additive process with triplet $(0, \beta, 0)$, that is, a centered Gaussian process with variance process $\sigma^2 = \{\sigma_t^2, t \geq 0\}$.

Recall that function β is continuous, null at the origin and increasing. It can be seen as a diffuse and $\sigma-$finite measure on $[0, \infty)$. The case β is the Lebesgue measure corresponds with the case W is the standard Brownian motion. Note also that if $\bar{W}$ denotes a standard Brownian motion we can write $W_t = \sqrt{\beta(t)}\bar{W}_t$.

Therefore, W is a continuous and centered Gaussian process defined on the complete probability space $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W)$ introduced in previous lecture. Denote by $\mathbb{E}_W$ the expectation with respect to $\mathbb{P}^W$.

The Gaussian case II

Recall that $\Omega^W = C_0([0,\infty))$, $\mathcal{F}^W$ is the $\sigma-$algebra generated by the topology of the uniform convergence on compact sets and $\mathbb{P}^W$ is the Wiener measure. Recall also that $\mathcal{F}^W$ coincides with the $\sigma-$algebra generated by the cylinder sets.

Given $C_1, \ldots, C_k$, subsets of $\mathbb{R}$ we have

$$\mathbb{P}(W(t_1) \in C_1, \ldots, W(t_k) \in C_k) = \int_{C_1 \times \cdots \times C_k} \prod_{i=1}^{k} p_i(x_{i-1}, x_i) dx_1 \cdots dx_k$$

with

$$p_i(x_{i-1}, x_i) := (2\pi(\beta(t_i) - \beta(t_{i-1})))^{-\frac{1}{2}} \exp\{-\frac{(x_i - x_{i-1})^2}{2(\beta(t_i) - \beta(t_{i-1}))}\}.$$

The Gaussian case III

Also from previous lecture we know that process W enjoys the chaotic representation property and therefore we can apply the machinery developed in Lecture 2 to this case.

So, the purpose of this subsection is to give the probabilistic interpretation of operators D and δ in the Gaussian setting.

Recall that we are in a particular case of the framework of previous lecture, with $M = W$, a centered independent random measure on $[0, \infty)$ with variance or intensity β. The multiple stochastic integrals are integrals with respect W.

First of all we present the theory for functionals of process W taking values in $\mathbb{R}$ and later we extend it to functionals taking values in a Hilbert space. This extension will be necessary for our purposes as we have commented in the previous section.

Probabilistic interpretation of the gradient operator I

We introduce now the probabilistic interpretation of operator D.

Let F be a random variable defined on Ω^W. Take $g \in L^2([0,\infty),\beta)$. Define

$$\gamma(t) := \int_0^t g(s)\beta(ds)$$

Note that γ is a well defined continuous function, null at the origin, and so, it belongs to Ω^W. Then, we can define the directional derivative of F in the direction of γ as

$$(D_\gamma F)(\omega) := \frac{d}{d\epsilon}(F(\omega+\epsilon\gamma))(0) = \lim_{\epsilon\downarrow 0}\frac{F(\omega+\epsilon\gamma)-F(\omega)}{\epsilon}.$$

With this definition we are defining (Gateaux type) derivatives of F in the directions defined by functions γ, but not necessary in all possible directions in the Banach space Ω^W. The space of all functions γ endowed with the norm $||\gamma||_{CM} := ||g||_{L^2}$ is called the Cameron-Martin space and denoted by CM.

Probabilistic interpretation of the gradient operator II

Definition: Given a random variable F, if $D_\gamma F$ exists in $L^2(\Omega^W)$ for any $\gamma \in CM$ and moreover it exists a process $\psi(\omega, t) \in L^2(\Omega^W \times [0, \infty))$ such that

$$D_\gamma F(\omega) = \int_0^\infty \psi(\omega, t) g(t) \beta(dt),$$

for any $g \in L^2([0, \infty), \beta)$, we say that F is stochastically differentiable and define its stochastic derivative as

$$\mathcal{D}_t F(\omega) := \psi(\omega, t).$$

Probabilistic interpretation of the gradient operator III

Consider now the set $\mathcal{P}_n$ of polynomials on $\mathbb{R}^n$. Denote by p an element of $\mathcal{P}$. Given n and $f_1, \ldots, f_n$, functions of $L^2([0,\infty), \beta)$, we denote by $\mathcal{S}$ the class of functionals

$$F := p(I_1(f_1), \ldots, I_1(f_n)).$$

We can see that these functionals are stochastically differentiable.

We have the following chain rule:

Proposition: The functional $F := p(I_1(f_1), \ldots, I_1(f_n))$ is stochastically differentiable and

$$\mathcal{D}_t F = \sum_{i=1}^{n} (\partial_i p)(I_1(f_1), \ldots, I_1(f_n)) f_i(t).$$

Probabilistic interpretation of the gradient operator IV

Proof Take γ with kernel $g \in L^2([0,\infty),\beta)$. Then,

$$
\begin{aligned}
& \frac{d}{d\epsilon} p(I_1(f_1) + \epsilon \int_0^\infty f_1(s) g(s) \beta(ds), \ldots, I_1(f_n) + \epsilon \int_0^\infty f_n(s) g(s) \beta(ds)) \\
= & \sum_{i=1}^n \partial_i p(I_1(f_1), \ldots, I_1(f_n)) \frac{d}{d\epsilon} (I_1(f_i) + \epsilon \int_0^\infty f_i(s) g(s) \beta(ds)) \\
= & \sum_{i=1}^n \partial_i p(I_1(f_1), \ldots, I_1(f_n)) \int_0^\infty f_i(s) g(s) \beta(ds)) \\
= & \int_0^\infty (\sum_{i=1}^n \partial_i p(I_1(f_1), \ldots, I_1(f_n)) f_i(s)) g(s) \beta(ds)).
\end{aligned}
$$

So,

$$
\mathcal{D}_t F = \sum_{i=1}^n (\partial_i p)(I_1(f_1), \ldots, I_1(f_n)) f_i(t).
$$

Probabilistic interpretation of the gradient operator V

Corollary: The functional $F := p(W(t_1), \dots, W(t_n))$ is stochastically differentiable and

$$\mathcal{D}_t F = \sum_{i=1}^{n} (\partial_i p)(W(t_1), \dots, W(t_n)) \mathbf{1}_{[0,t_i]}(t).$$

Proof: The proof is straightforward from the previous proposition taking $f_i = \mathbf{1}_{[0,t_i]}$ for $i = 1, \dots, n$.

Given $F \in \mathcal{S}$ we can consider the norm

$$||F||_{1,2} = \{\mathbb{E}(|F|^2) + \mathbb{E}(||\mathcal{D}F||_2^2)\}^{\frac{1}{2}}$$

and define the domain $\mathcal{D}^{1,2}$ as the closure of $\mathcal{S}$ with respect to this norm.

Probabilistic interpretation of the gradient operator VI

The following lemmas ensures the operator $\mathcal{D}$ defined on $\mathcal{S}$ is closed.

The first one is a integration by parts formula in $\mathcal{D}^{1,2}$

Lemma: Consider $F, G \in \mathcal{D}^{1,2}$. Let $\gamma \in CM$ with kernel $h \in L^2(\beta)$. Then,

$$\mathbb{E}[D_\gamma F \cdot G] = \mathbb{E}[FG \int_0^\infty h(s) dW_s] - \mathbb{E}[F \cdot D_\gamma G].$$

Proof: See Di Nunno-Oksendal-Proske (2007).

Probabilistic interpretation of the gradient operator VII

Lemma: Let $\{F_n, n \geq 1\} \subseteq \mathcal{S}$ such that converges to 0 in $L^2(\Omega)$. Assume moreover that $\{\mathcal{D}F_n, n \geq 1\}$ converges in $L^2(\Omega \times [0,\infty))$. Therefore,

$$\lim_{n\uparrow\infty} \mathcal{D}_t F_n = 0.$$

Proof: Given any $G \in \mathcal{S}$ and $\gamma \in CM$ with kernel $h \in L^2(\beta)$ we have

$$\mathbb{E}[D_\gamma F_n \cdot G] = \mathbb{E}[F_n G \int_0^\infty h(s) dW_s] - \mathbb{E}[F_n D_\gamma G].$$

By hypothesis, the right hand side converges to 0. Therefore, the left hand side converges to 0 for any $G \in \mathcal{S}$. Being $\mathcal{S}$ dense and using the hypothesis that $D_\gamma F_n$ converges in $L^2(\Omega^W)$ we conclude $D_\gamma F_n$ converges to 0. Finally, since it holds for any γ and $h \in L^2(\beta)$ we obtain the result.

Probabilistic interpretation of the gradient operator VIII

Therefore, if $F \in \mathcal{D}^{1,2}$, there exists a sequence $\{F_n, n \geq 1\} \subseteq \mathcal{S}$ such that $F = \lim_{n\uparrow\infty} F_n$ in $L^2(\Omega^W)$ and $\mathcal{D}F_n$ converges in $L^2(\Omega^W \times [0,\infty))$. Then, we can define

$$\mathcal{D}F := \lim_{n\uparrow\infty} \mathcal{D}F_n$$

and

$$\mathcal{D}_\gamma F = \int_0^\infty \mathcal{D}_s F \, g(s)\beta(ds).$$

Probabilistic interpretation of the gradient operator IX

Note that the stochastic derivative of F is an element of $L^2(\Omega^W \times [0,\infty))$.

The purpose now is to show that $\mathbb{D}^{1,2} \subseteq \mathcal{D}^{1,2}$ and for $F \in \mathbb{D}^{1,2}$ we have $D_t F = \mathcal{D}_t F$.

In this sense $\mathcal{D}$, the stochastic derivative defined on the canonical space of the Gaussian additive process Ω^W coincide with the annihilation operator D defined before, and so, is its probabilistic interpretation.

Probabilistic interpretation of the gradient operator X

Note first of all that the two operators coincide on random variables of the first chaos.

Take $F := \int_0^\infty f(s)dW(s)$ with $f \in L^2(\beta)$. Take $\gamma \in CM$ with kernel $g \in L^2(\beta)$. Then,

$$F(\omega + \epsilon\gamma) = (\int_0^\infty f(s)dW(s))(\omega + \epsilon\gamma) = \int_0^\infty f(s)d\omega(s) + \epsilon \int_0^\infty f(s)g(s)\beta(ds),$$

and the derivative with respect to ϵ evaluated at $\epsilon = 0$ is equal to

$$\int_0^\infty f(s)g(s)\beta(ds)$$

So, seeing the corresponding definitions, we have that $F \in \mathcal{D}^{1,2}$ and $\mathcal{D}_t F = f(t)$, for any $t \geq 0$ and $\omega \in \Omega^W$. Note that this coincide with the annihilation operator $D_t F$. Note in particular that if $f := \mathbf{1}_{[0,u]}$, we have

$$\mathcal{D}_t W(u) = \mathbf{1}_{[0,u]}(t).$$

Probabilistic interpretation of the gradient operator XI

The following proposition shows the definition given above coincide with the annihilation operator introduced in Lecture 2.

Proposition: Let $F \in \mathbb{D}_W^{1,2}$ with chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in L_s^2([0,\infty)^n)$, for all $n \geq 0$. Then, $F \in \mathcal{D}^{1,2}$ and

$$\mathcal{D}_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)).$$

Therefore $\mathcal{D}_t F = D_t F$ and $\mathcal{D}$ is the probabilistic interpretation of D in the canonical space.

Probabilistic interpretation of the gradient operator XII

Proof: By linearity and denseness it is sufficient to prove

$$\mathcal{D}_t I_m(f_m) = m I_{m-1}(f_m(\cdot, t)).$$

For

$$f_m(t_1, \ldots, t_m) = \mathbf{1}_{A_1 \times \cdots \times A_m}(t_1, \ldots, t_m),$$

where $A_1, \ldots, A_m$ are pairwise disjoint Borel sets with $\beta(B) < \infty$, we have

$$\begin{aligned} \mathcal{D}_t(I_m(f_m)) &= \mathcal{D}_t(W(A_1) \cdots W(A_m)) \\ &= \sum_{j=1}^{m} W(A_1) \cdots W(A_{j-1}) W(A_{j+1}) \cdots W(A_m) \mathbf{1}_{A_j}(t) \\ &= m I_{m-1}(f_m(\cdot, t)). \end{aligned}$$

Chain rule I

A very useful rule to manage the Malliavin derivative is the following chain rule.

Proposition: Let φ a Lipschitz function on $\mathbb{R}^m$ with Lipschitz constant $K > 0$. Let $F = (F^{(1)}, \ldots, F^{(m)})$ a random vector such that $F^{(i)} \in \mathbb{D}^{1,2}$ for any $i = 1, \ldots, m$. Then, $\varphi(F) \in \mathbb{D}^{1,m}$ and there exists a random vector $G = (G^{(1)}, \ldots, G^{(m)})$ such that $|G^{(i)}| \leq K$ for any $i = 1, \ldots, m$ and

$$D(\varphi(F)) = \sum_{i=1}^{m} G_i DF^{(i)}.$$

Proof: Nualart (2006)

Chain rule II

A particular case is when function φ is continuously differentiable with bounded partial derivatives. In this case we have the following result

Proposition: Let φ a function of $C_b^1(\mathbb{R}^m)$. Let $F = (F^{(1)}, \ldots, F^{(m)})$ a random vector such that $F^{(i)} \in \mathbb{D}^{1,2}$ for any $i = 1, \ldots, m$. Then, $\varphi(F) \in \mathbb{D}^{1,m}$ and

$$D(\varphi(F)) = \sum_{i=1}^{m} (\partial_i \varphi)(F) DF^{(i)}.$$

Proof: Nualart (2006)

Product rule

An important corollary is the following

Corollary: Let F and G two random variables of $L^2(\Omega)$. Assume one of the two following conditions is satisfied.

i) $F, G \in \mathbb{D}^{1,2}$ and F and $||DF||_H$ are bounded.
ii) F and G have a chaos expansion with a finite number of terms.

Then, we have the following product rule

$$D(FG) = FDG + GDF.$$

As we will see, the previous properties are specific of the Gaussian setting. The formulas in the pure jump setting will be different. For example the product rule is different.

Probabilistic interpretation of the divergence operator I

Let us study now the probabilistic interpretation of operator δ. The main result is the fact that for adapted processes, the operator δ coincides with the Itô integral.

Let $u \in L^2(\Omega \times [0,\infty))$. We know we can represent process u as

$$u(\theta) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,\theta))$$

with $f_n \in L^2([0,\infty)^{n+1})$ and symmetric with the first n variables. Denote by $\tilde{f}_n$ the symmetrization of f_n.

The domain $Dom\delta$ is the subspace of processes of $L^2(\Omega \times [0,\infty))$ such that

$$\sum_{n=0}^{\infty} (n+1)! ||\tilde{f}_n||^2_{L^2_s([0,\infty)^{n+1})} < \infty. \tag{4}$$

Probabilistic interpretation of the divergence operator II

For processes of $Dom\delta$ we can define

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

that clearly is a well-defined sum in $L^2(\Omega)$.

Note that δ is a linear operator from $L^2(\Omega \times [0,\infty)$ to $L^2(\Omega)$. It is immediate to see that $\mathbb{E}(\delta(u)) = 0$ because multiple stochastic integrals have zero expectation except in the case $n = 0$ that not appears in the expansion of $\delta(u)$.

As we have seen before, $u \in L^2(\Omega \times [0,\infty))$ is adapted if and only if $f_n(t_1, \ldots, t_n, t) = 0$ for all $n \geq 1$ and $t < \max_{1 \leq i \leq n} t_i$.

Probabilistic interpretation of the divergence operator III

The following result characterizes the operator δ as an integral. For adapted processes it is a generalization of the Itô integral.

Proposition: Let $u \in L^2(\Omega \times [0,\infty))$ an adapted process. Then, it belongs to $Dom\delta$ and

$$\delta(u) = \int_0^\infty u(s) dW_s.$$

Proof: Nualart (2006).

Probabilistic interpretation of the divergence operator IV

Finally, we can see the Skorohod integral δ as a stochastic process defining

$$\delta_t^W(u) := \delta^W(u\mathbf{1}_{[0,t]}).$$

The following formula is very useful and it is also specific of the Gaussian setting.

Proposition: Assume $F \in DomD^W$ and $u \in Dom\delta^W$. Assume moreover $F \cdot u \in L^2(\Omega \times [0,\infty))$. Then, $F \cdot u \in Dom\delta$ and

$$\delta^W(F \cdot u) = F\delta^W(u) - \int_0^\infty u_t D_t^W F \beta(dt)$$

provided one of the two sides of the equality belongs to $L^2(\Omega)$.

Probabilistic interpretation of the divergence operator V

We define the space $\mathbb{L}^{1,2}_W := L^2([0,\infty); DomD^W)$, that is the space of processes $u \in L^2([0,\infty) \times \Omega^W)$ such that $u_t \in DomD^W$, for almost all t, and $Du \in L^2(\Omega^W \times [0,\infty)^2)$.

It can be proved that $\mathbb{L}^{1,2}_W \subseteq Dom\delta^W$ and

$$\mathbb{E}_W(\delta^W(u)^2) \leq ||u||^2_{\mathbb{L}^{1,2}_W} := \mathbb{E}_W(||u||^2_{L^2([0,\infty))}) + \mathbb{E}_W(||D^W u||^2_{L^2([0,\infty)^2)}).$$

Extension to Hilbert space values random variables I

As we have commented before, the operator D can be extended to random variables taking values in a Hilbert space H.

Let $\mathcal{S}^{W,H}$ the set of smooth $H-$valued random variables of the form

$$F = \sum_{i=1}^{n} G_i h_i$$

where $h_i \in H$ and G_i is a smooth random variable of type $f(W(h_1), \ldots, W(h_n))$ with $f \in \mathcal{S}_p$.

Extension to Hilbert space values random variables II

Then, we define

$$\tilde{D}F = \sum_{i=1}^{n} DG_i \otimes h_i$$

and denote by $Dom\tilde{D}$ the completion of $\mathcal{S}^{W,H}$ by the norm defined by

$$||F||^2_{W,H} := \mathbb{E}(||F||^2_H) + \mathbb{E}\int_0^\infty ||\tilde{D}_t F||^2_H \beta(dt).$$

As we have seen before, this will be useful to interpret the operator $D_{t,0}$.

The pure jump case

In this section we establish the operators and the basic calculus rules of a Malliavin-Skorohod calculus with respect to a pure jump additive process on the canonical space defined before. I follow essentially Di Nunno - Vives (2017).

Now X denotes an additive process of triplet $(0, 0, \nu_t)$ and the process is determined by the associated measure ν defined on $\Theta_{\infty,0}$.

Basic canonical transformations

Let $\theta = (s, x) \in \Theta_{\infty,0}$. Let $\omega \in \Omega^J$, that is, $\omega := (\theta_1, \ldots, \theta_n, \ldots)$, with $\theta_i := (s_i, x_i)$.

We introduce the following two transformations from $\Theta_{\infty,0} \times \Omega^J$ to Ω^J :

$$\epsilon_\theta^+ \omega := \big((s, x), (s_1, x_1), (s_2, x_2), \ldots\big),$$

where a jump of size x is added at time s, and

$$\epsilon_\theta^- \omega := \big((s_1, x_1), (s_2, x_2), \ldots\big) - \{(s, x)\},$$

where we take away the point $\theta = (s, x)$ from ω.

Observe that ϵ^+ is well defined on Ω^J except on the set $\{(\theta, \omega) : \theta \in \omega\}$, which has null $\nu \otimes \mathbb{P}$ measure. We can set by convention that on this set, $\epsilon_\theta^+ \omega := \omega$. The case of ϵ_θ^- is also clear. In fact this operator satisfies $\epsilon_\theta^- \omega = \omega$ except on the set $\{(\theta, \omega) : \theta \in \omega\}$. For simplicity of the notation, when needed, we will denote $\hat{\omega}_i := \epsilon_{\theta_i}^- \omega$.

Basic canonical operators I

Two previous transformations are analogous to the ones introduced in Picard (1996), where they are called creation and annihilation operators. Here, we develop the theory constructively on the canonical space. See Meyer (1993) for general information about creation and annihilation operators in quantum probability.

Let $L^0(\Omega^J)$ denote the set of random variables defined on Ω^J and by $L^0(\Theta_{\infty,0} \times \Omega^J)$ the set of measurable stochastic processes defined on $\Theta_{\infty,0} \times \Omega^J$. Now we consider the following two definitions:

Definition: For a random variable $F \in L^0(\Omega^J)$, we define the operator

$$\mathcal{T} : L^0(\Omega^J) \longmapsto L^0(\Theta_{\infty,0} \times \Omega^J),$$

such that $(\mathcal{T}_\theta F)(\omega) := F(\epsilon_\theta^+ \omega)$.

Basic canonical operators II

If F is a $\mathcal{F}^J$-measurable, then

$$(\mathcal{T}.F)(\cdot)\colon \Theta_{\infty,0} \times \Omega^J \longrightarrow \mathbb{R}$$

is $\mathcal{B}(\Theta_{\infty,0}) \otimes \mathcal{F}^J -$ measurable.

Moreover, it $F = 0$, $\mathbb{P}$-a.s., then $\mathcal{T}.F(\cdot) = 0$, $\nu \otimes \mathbb{P}$-a.e.

So, $\mathcal{T}$ is a closed linear operator defined on the entire $L^0(\Omega^J)$. See Solé-Utzet-Vives (2007).

From now on we use for any $m > 0$ the notation $\Theta_m = [0, m] \times \{|x| > \frac{1}{m}\} \subseteq \Theta_{\infty,0}$.

Basic canonical operators III

If we want to secure $\mathcal{T}.F(\cdot) \in L^1(\Theta_{\infty,0} \times \Omega^J)$, we have to restrict the domain and guarantee that

$$\mathbb{E} \int_{\Theta_{\infty,0}} |\mathcal{T}_\theta F| \nu(d\theta) < \infty.$$

Remark that this requires a condition that is strictly stronger than $F \in L^1(\Omega^J)$. Concretely, we have to assume that

$$\sum_{m=1}^{\infty} e^{-\nu(\Theta_m - \Theta_{m-1})} \sum_{n=0}^{\infty} \frac{n}{n!} \int_{(\Theta_m - \Theta_{m-1})^n} |F(\theta_1, \ldots, \theta_n)| \nu(d\theta_1) \ldots \nu(d\theta_n) < \infty,$$

whereas $F \in L^1(\Omega)$ is equivalent only to

$$\sum_{m=1}^{\infty} e^{-\nu(\Theta_m - \Theta_{m-1})} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Theta_m - \Theta_{m-1})^n} |F(\theta_1, \ldots, \theta_n)| \nu(d\theta_1) \ldots \nu(d\theta_n) < \infty.$$

Basic canonical operators IV

Definition: For a random field $u \in L^0(\Theta_{\infty,0} \times \Omega^J)$, we define the operator

$$\mathcal{S} : DomS \subseteq L^0(\Theta_{\infty,0} \times \Omega^J) \longrightarrow L^0(\Omega^J)$$

such that

$$(\mathcal{S}u)(\omega) := \int_{\Theta_{\infty,0}} u_\theta(\epsilon_\theta^- \omega) N(d\theta, \omega) := \sum_i u_{\theta_i}(\hat{\omega}_i) < \infty.$$

In particular, if $\omega = \alpha$, we define $(\mathcal{S}u)(\alpha) := 0$.

The operator $\mathcal{S}$ is well defined on $L^1(\Theta_{\infty,0} \times \Omega^J)$ as the following proposition says:

Proposition: If $u \in L^1(\Theta_{\infty,0} \times \Omega^J)$, $\mathcal{S}u$ is well defined and takes values in $L^1(\Omega)$. Moreover

$$\mathbb{E} \int_{\Theta_{\infty,0}} u_\theta(\epsilon_\theta^- \omega) N(d\theta, \omega) = \mathbb{E} \int_{\Theta_{\infty,0}} u_\theta(\omega) \nu(d\theta). \quad (5)$$

Basic canonical operators V

Proof: Fix Ω_m^J and denote, for any $n \geq 0$, $\omega := (\theta_1, \ldots, \theta_n)$ and $\theta := (s, x)$. Denote also $c_m := e^{-\nu(\Theta_m)}$. We have

$$\mathbb{E}(\mathbf{1}_{\Omega_m^J} \int_{\Theta_m} u_\theta(\epsilon_\theta^- \omega) N(d\theta, \omega)) = \sum_{n=1}^{\infty} \frac{c_m}{n!} \int_{\Theta_m^n} \sum_{i=1}^{n} u_{\theta_i}(\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_n) \nu(d\theta_1) \cdots \nu(d\theta_n)$$

$$= \sum_{n=1}^{\infty} \frac{c_m}{n!} \int_{\Theta_m^n} n u_\theta(\theta_1, \ldots, \theta_{n-1}) \nu(d\theta_1) \cdots \nu(d\theta_{n-1}) \nu(d\theta)$$

$$= \sum_{n=1}^{\infty} \frac{c_m}{(n-1)!} \int_{\Theta_m^{n-1}} \int_{\Theta_m} u_\theta(\theta_1, \ldots, \theta_{n-1}) \nu(d\theta_1) \cdots \nu(d\theta_{n-1}) \nu(d\theta)$$

$$= \sum_{l=0}^{\infty} \frac{c_m}{l!} \int_{\Theta_m^l} \int_{\Theta_m} u_\theta(\theta_1, \ldots, \theta_l) \nu(d\theta_1) \cdots \nu(d\theta_l) \nu(d\theta) = \mathbb{E}(\mathbf{1}_{\Omega_m^J} \int_{\Theta_m} u_\theta \nu(d\theta))$$

The general case comes from dominated convergence.

Basic canonical operators VI

Remark: Formula (5) is a version, in our canonical space, of the so-called Mecke formula. See, for example, Peccati-Reitzner (2016)..

Remark: We have proved that $L^1(\Theta_{\infty,0} \times \Omega^J) \subseteq Dom\mathcal{S}$. Moreover $\mathcal{S}$ is closed in L^1 as an operator from $L^1(\Theta_{\infty,0} \times \Omega^J)$ to $L^1(\Omega)$. In fact, if we take a sequence $u^{(n)} \in L^1(\Theta_{\infty,0} \times \Omega^J)$ converging to 0 in this space and we assume that $Su^{(n)}$ converges to G in $L^1(\Omega^J)$, we can show that $G = 0$. This is immediate because

$$\mathbb{E}|G| \leq \mathbb{E}|G - \mathcal{S}u^{(n)}| + \mathbb{E}|\mathcal{S}u^{(n)}|.$$

Indeed, the first term in the right hand side converges to 0 by hypothesis and the second one can be bounded by

$$\mathbb{E}|\mathcal{S}u^{(n)}| \leq \mathbb{E}\int_{\Theta_{\infty,0}} |u_\theta^{(n)}(\epsilon_\theta^- \omega)| N(d\theta, \omega) = \mathbb{E}\int_{\Theta_{\infty,0}} |u_\theta^{(n)}| \nu(d\theta),$$

which also converges to 0 by hypothesis.

Basic canonical operators VII

Remark: Given $\theta = (s, x)$, for any ω, we can define $\tilde{\omega}_s$ as the restriction of ω to jump instants strictly before s. In this case, obviously, $\epsilon_\theta^- \tilde{\omega}_s = \tilde{\omega}_s$. If u is predictable we have $u_\theta(\omega) = u_\theta(\tilde{\omega}_s)$. In this case, we have

$$u_\theta(\epsilon_\theta^- \omega) = u_\theta((\widetilde{\epsilon_\theta^- \omega})_s) = u_\theta(\tilde{\omega}_s) = u_\theta(\omega),$$

and

$$(\mathcal{S}u)(\omega) = \int_{\Theta_{\infty,0}} u_\theta(\epsilon_\theta^- \omega) N(d\theta, \omega) = \int_{\Theta_{\infty,0}} u_\theta(\omega) N(d\theta, \omega).$$

Basic duality formula I

Hereafter we introduce a fundamental relationship between the two operators $\mathcal{S}$ and $\mathcal{T}$:

Theorem: Consider $F \in L^0(\Omega^J)$ and $u \in DomS$. Then $F \cdot \mathcal{S}u \in L^1(\Omega^J)$ if and only if $\mathcal{T}F \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$ and in this case

$$\mathbb{E}(F \cdot \mathcal{S}u) = \mathbb{E} \int_{\Theta_{\infty,0}} \mathcal{T}_\theta F \cdot u_\theta \, \nu(d\theta).$$

Basic duality formula II

Proof: Using the fact that F is symmetric, we have

$$
\begin{aligned}
\mathbb{E}(F \cdot \mathcal{S}u \cdot \mathbf{1}_{\Omega_m^J}) &= \sum_{n=1}^{\infty} \frac{c_m}{n!} \int_{\Theta_m^n} F(\theta_1, \ldots \theta_n)(Su)(\theta_1, \ldots, \theta_n)\nu(d\theta_1)\cdots\nu(d\theta_n) \\
&= \sum_{n=1}^{\infty} \frac{c_m}{n!} \int_{\Theta_m^n} F(\theta_1, \ldots \theta_n) \sum_{i=1}^{n} u_{\theta_i}(\hat{\omega}_i)\nu(d\theta_1)\cdots\nu(d\theta_n) \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{c_m}{n!} \int_{\Theta_m^n} \mathcal{T}_{\theta_i} F(\theta_1, \ldots, \hat{\theta}_i, \ldots \theta_n) u_{\theta_i}(\hat{\omega}_i)\nu(d\theta_1)\cdots\nu(d\theta_n) \\
&= \sum_{n=1}^{\infty} n \frac{c_m}{n!} \int_{\Theta_m^{n-1}} \int_{\Theta_m} \mathcal{T}_{\theta} F(\theta_1, \ldots \theta_{n-1}) u_{\theta}(\hat{\omega}_n)\nu(d\theta_1)\cdots\nu(d\theta_{n-1})\nu(d\theta) \\
&= \mathbb{E}\left(\mathbf{1}_{\Omega_m^J} \int_{\Theta_m} \mathcal{T}_{\theta} F u_{\theta}\, \nu(d\theta)\right)
\end{aligned}
$$

Finally, we extend the result to Ω^J using the dominated convergence theorem.

Basic rules of calculus I

Moreover we obtain the following rules of calculus:

Proposition: If u and $\mathcal{T}F \cdot u$ belong to $Dom\mathcal{S}$, then we have $F \cdot \mathcal{S}u = \mathcal{S}(\mathcal{T}F \cdot u)$, $\mathbb{P} - a.e.$

Proof: This is an immediate consequence of the fact that $\mathcal{T}_{\theta_i} F(\hat{\omega}_i) = F(\omega)$.

Basic rules of calculus II

Proposition: If u and $\mathcal{T}u$ are in $Dom\mathcal{S}$, then $\mathcal{T}_\theta(\mathcal{S}u) = u_\theta + \mathcal{S}(\mathcal{T}_\theta u)$, $\nu \otimes \mathbb{P} - a.e.$

Proof: For the left-hand side term, we have

$$\mathcal{T}_\theta(\mathcal{S}u)(\omega) = (\mathcal{S}u)(\epsilon_\theta^+\omega) = u_\theta(\omega) + \sum_i u_{\theta_i}(\epsilon_{\theta_i}^-\epsilon_\theta^+\omega)$$

and, for the right-hand side term, we have

$$u_\theta(\omega) + \mathcal{S}(\mathcal{T}_\theta u)(\omega) = u_\theta(\omega) + \sum_i u_{\theta_i}(\epsilon_\theta^+\epsilon_{\theta_i}^-\omega).$$

The equality comes from $\epsilon_{\theta_i}^-\epsilon_\theta^+\omega = \epsilon_\theta^+\epsilon_{\theta_i}^-\omega$, $\nu \otimes \mathbb{P} - a.e.$

Intrinsic operators I

With the results of the previous sections we are ready to introduce two operators which also turn out to fulfill a duality relationship.

These operators will be hereafter called intrinsic operators, being defined constructively on the canonical space.

We define the operator
$$\Psi_\theta := \mathcal{T}_\theta - Id.$$
Observe that this operator is linear, closed and satisfies the property

$$\Psi_\theta(F\,G) = G\,\Psi_\theta F + F\,\Psi_\theta G + \Psi_\theta(F)\,\Psi_\theta(G).$$

Intrinsic operators II

On other hand, for $u \in L^0(\Theta_{\infty,0} \times \Omega^J)$, we consider the operator:

$$\mathcal{E} : Dom\mathcal{E} \subseteq L^0(\Theta_{\infty,0} \times \Omega^J) \longrightarrow L^0(\Omega^J)$$

such that

$$(\mathcal{E}u)(\omega) := \int_{\Theta_{\infty,0}} u_\theta(\omega)\nu(d\theta).$$

Note that $Dom\mathcal{E}$ is the subset of processes in $L^0(\Theta_{\infty,0} \times \Omega^J)$ such that $u(\cdot,\omega) \in L^1(\Theta_{\infty,0})$, $\mathbb{P}-$a.s. On other hand recall that, for ω fixed, we have $\epsilon_\theta^- \omega = \omega$, if $\theta \neq \theta_i$, for any i, and that $\nu(\{\theta : \theta = \theta_i, \text{ for some } i\}) = 0$. So,

$$\int_{\Theta_{\infty,0}} u_\theta(\epsilon_\theta^- \omega)\nu(d\theta) = \int_{\Theta_{\infty,0}} u_\theta(\omega)\nu(d\theta), \quad \mathbb{P} - a.s. \tag{6}$$

Intrinsic operators III

Then, for $u \in Dom\Phi := Dom\mathcal{S} \cap Dom\mathcal{E} \subseteq L^0(\Theta_{\infty,0} \times \Omega^J)$, we define

$$\Phi u := \mathcal{S}u - \mathcal{E}u.$$

Remarks:

a) Observe that $L^1(\Theta_{\infty,0} \times \Omega^J) \subseteq Dom\Phi$.

b) Note that from Mecke formula and (6), we have that $E(\Phi u) = 0$, for any $u \in L^1(\Theta_{\infty,0} \times \Omega)$.

c) Moreover, for any predictable $u \in Dom\Phi$ we have

$$\Phi(u) = \int_{\Theta_{\infty,0}} u_\theta(\omega)\tilde{N}(d\theta, \omega),$$

Intrinsic operators IV

As a corollary of the basic duality formula we have the following result:

Proposition: Consider $F \in L^0(\Omega^J)$ and $u \in Dom\Phi$. Assume also $F \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$. Then $F \cdot \Phi u \in L^1(\Omega^J)$ if and only if $\Psi F \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$ and in this case

$$\mathbb{E}(F \cdot \Phi u) = \mathbb{E}\Big(\int_{\Theta_{\infty,0}} \Psi_\theta F \cdot u_\theta \, \nu(d\theta) \Big).$$

Intrinsic operators V

Analogously to the previous subsection we have also the following two results that can be proved immediately from previous results and recalling the definitions $\Psi = \mathcal{T} - Id$ and $\Phi = \mathcal{S} - \mathcal{E}$.

Proposition: If $F \in L^0(\Omega^J)$ and u, $F \cdot u$ and $\Psi F \cdot u$ belong to $Dom\Phi$, then we have

$$F \cdot \Phi u = \Phi(F \cdot u) + \Phi(\Psi F \cdot u) + \mathcal{E}(\Psi F \cdot u), \quad \mathbb{P} - a.s.$$

Proposition: If u and Ψu belong to $Dom\Phi$, we have

$$\Psi_\theta(\Phi u) = u_\theta + \Phi(\Psi_\theta u), \quad \nu \otimes \mathbb{P} - a.e.$$

Intrinsic operators VI

If we change $\nu(ds, dx)$ by $x^2\nu(ds, dx)$ and we define the operators

$$\bar{\Psi}_{s,x}F := \frac{\mathcal{T}_{s,x}F - F}{x},$$

$$\bar{\mathcal{S}}u(\omega) := \int_{\Theta_{\infty,0}} u_{s,x}(\epsilon^{-}_{s,x}\omega)xN(ds, dx),$$

$$(\bar{\mathcal{E}}u)(\omega) := \int_{\Theta_{\infty,0}} u_{s,x}(\omega)x^2\nu(ds, dx)$$

and

$$\bar{\Phi} := \bar{\mathcal{S}} - \bar{\mathcal{E}},$$

we can prove similar results to the previous ones.

Intrinsic operators VII

For example, if $F \in L^0(\Omega^J)$, $u \in Dom\bar{\Phi}$, and $F \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$, then $F \cdot \bar{\Phi} u \in L^1(\Omega^J)$ if and only if $\bar{\Psi} F \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$ and in this case

$$\mathbb{E}(F \cdot \bar{\Phi} u) = \mathbb{E}\Big(\int_{\Theta_{\infty,0}} \bar{\Psi}_{s,x} F \cdot u_{s,x}\, x^2 \nu(ds,dx) \Big).$$

Note that the domains of $\bar{\Psi}$ and Ψ are slightly different in view of the use of a different measure. This has natural consequences also on the evaluations in L^1. For example,

$$\mathbb{E} \int_{\Theta_{\infty,0}} |\bar{\Psi}_{s,x} F| x^2 \nu(ds,dx) = \mathbb{E} \int_{\Theta_{\infty,0}} |\Psi_{s,x} F||x| \nu(ds,dx) \neq \mathbb{E} \int_{\Theta_{\infty,0}} |\Psi_{s,x} F| \nu(ds,dx).$$

Intrinsic operators and multiple stochastic integrals I

In the last part of this section we study the intrinsic operators Ψ and Φ in comparison with the Malliavin derivative and Skorohod integral D^J and δ^J defined before restricted to the pure jump case, i.e. associated with $\tilde{N}(ds, dx)$.

First we need to recall some preliminary results.

The following key lemma is proved in Solé-Utzet-Vives (2007) and it is an extension of Lemma 2 in Nualart-Vives (1995).

Intrinsic operators and multiple stochastic integrals II

Lemma: For any $n \geq 1$, consider the set

$$\Theta_{T,\epsilon}^{n,*} = \{(\theta_1, \dots, \theta_n) \in \Theta_{T,\epsilon}^{n} : \theta_i \neq \theta_j \text{ if } i \neq j\}.$$

Then, for any $g_k \in L^2(\Theta_{\infty,0}^{k,*})$, $k \geq 1$, and $\omega \in \Omega^J$, we have

$$I_k(g_k)(\omega) = \int_{\Theta_{T,\epsilon}^{k,*}} g_k(\theta_1 \dots, \theta_k)\tilde{N}(\omega, d\theta_1) \cdots \tilde{N}(\omega, d\theta_k), \quad \mathbb{P} - a.e.$$

Proof: Both expressions coincide for simple functions and define bounded linear operators. We remark that g_k does not need to be symmetric.

Intrinsic operators and multiple stochastic integrals III

The relationships between D^J and Ψ, and δ^J and Φ are given by the following results, which extend corresponding results for the standard Poisson process given in Nualart and Vives (1995).

Lemma: For a fixed $k \geq 0$, consider $F = I_k(g_k)$ with g_k a symmetric function of $L^2(\Theta^{k,*}_{\infty,0})$. Then, F belongs to $Dom D^J \cap Dom \Psi$ and

$$D^J I_k(g_k) = \Psi I_k(g_k), \quad \nu \otimes \mathbb{P} - \text{a.e.}$$

Intrinsic operators and multiple stochastic integrals IV

Proof: The fact that $F \in DomD^J \cap Dom\Psi$ is obvious. From the definition of Ψ we obtain

$$\Psi_\theta I_k(g_k)(\omega) = I_k(g_k)(\epsilon_\theta^+\omega) - I_k(g_k)(\omega)$$

$$= \int_{\Theta_{\infty,0}^{k,*}} g_k(\theta_1,\ldots,\theta_k)\tilde{N}(\epsilon_\theta^+\omega, d\theta_1)\cdots\tilde{N}(\epsilon_\theta^+\omega, d\theta_k) - \int_{\Theta_{\infty,0}^{k,*}} g_k(\theta_1,\ldots,\theta_k)\tilde{N}(\omega, d\theta_1)\cdots\tilde{N}(\omega, d\theta_k$$

$$= \int_{\Theta_{\infty,0}^{k,*}} g_k(\theta_1,\ldots,\theta_k)\prod_{i=1}^{k}(\tilde{N}(\omega, d\theta_i) + N(\theta, d\theta_i)) - \int_{\Theta_{\infty,0}^{k,*}} g_k(\theta_1,\ldots,\theta_k)\prod_{i=1}^{k}\tilde{N}(\omega, d\theta_i).$$

Using the fact that g_k is null on the diagonals, only the integrals with $k-1$ integrators of type $\tilde{N}$ and one integrator of type N remain. Using the fact that g_k is symmetric in the last expression we obtain

$$\Psi_\theta I_k(g_k)(\omega) = k\int_{\Theta_{\infty,0}^{k-1,*}} g_k(\theta_1,\ldots,\theta_{k-1},\theta)\tilde{N}(\omega, d\theta_1)\cdots\tilde{N}(\omega, d\theta_{k-1}) = D_\theta^J I_k(g_k).$$

Intrinsic operators and multiple stochastic integrals V

Lemma: For fixed $k \geq 1$, consider $u_\theta = I_k(g_k(\cdot, \theta))$ where $g_k(\cdot, \cdot) \in L^2(\Theta_{\infty,0}^{k+1,*})$ is symmetric with respect to the first k variables. Assume also $u \in Dom\Phi$. Then,

$$\Phi(u) = \delta^J(u), \quad \mathbb{P} - \text{a.e..}$$

Intrinsic operators and multiple stochastic integrals VI

Proof: First of all, note that

$$\delta^J(I_k(g_k(\cdot,\theta))(\omega) = I_{k+1}(\tilde{g}_k(\cdot,\cdot))(\omega) = I_{k+1}(g_k(\cdot,\cdot))(\omega)$$

$$= \int_{\Theta_{\infty,0}^{k+1,*}} g_k(\theta_1,\ldots,\theta_k,\theta)\tilde{N}(\omega,d\theta_1)\cdots\tilde{N}(\omega,d\theta_k)N(\omega,d\theta) - \int_{\Theta_{\infty,0}} u_\theta \nu(d\theta)$$

$$= \sum_j \int_{\Theta_{\infty,0}^{k,*}} g_k(\theta_1,\ldots,\theta_k,\theta_j^0)\tilde{N}(\omega,d\theta_1)\cdots\tilde{N}(\omega,d\theta_k) - \int_{\Theta_{\infty,0}} u_\theta \nu(d\theta),$$

where the different θ_j^0 are the jump points of $\omega = (\theta_1^0, \theta_2^0, \ldots)$.
Recall that $\tilde{g}_k$, the symmetrization of g_k with respect to all its variables, is null on the diagonals, so θ_j^0 has to be different of all θ_i, for $i = 1, \ldots, k$.

Intrinsic operators and multiple stochastic integrals VII

Now observe that we can write $\tilde{N}(\omega, d\theta) = N(\theta_j^0, d\theta) + \tilde{N}(\epsilon^-_{\theta_j^0}\omega, d\theta)$, where, for simplicity, we write $\hat{\omega}_j := \epsilon^-_{\theta_j^0}\omega$. Then we have,

$$
\begin{aligned}
& \delta^J(u) \\
= \; & \sum_j \sum_{l=0}^{k} \binom{k}{l} \int_{\Theta^{k,*}_{\infty,0}} g_k(\theta_1, \ldots, \theta_k, \theta_j^0) N(\theta_j^0, d\theta_1) \cdots N(\theta_j^0, d\theta_l) \tilde{N}(\hat{\omega}_j, d\theta_{l+1}) \cdots \tilde{N}(\hat{\omega}_j, d\theta_k) \\
- \; & \int_{\Theta_{\infty,0}} u_\theta \nu(d\theta) \\
= \; & \sum_j \int_{\Theta^{k,*}_{\infty,0}} g_k(\theta_1, \ldots, \theta_k, \theta_j^0) \tilde{N}(\hat{\omega}_j, d\theta_1) \cdots \tilde{N}(\hat{\omega}_j, d\theta_k) - \int_{\Theta_{\infty,0}} u_\theta \nu(d\theta) = \Phi(u).
\end{aligned}
$$

Intrinsic operators and multiple stochastic integrals VIII

Remark: Recall that $u \in L^2(\Theta_{\infty,0} \times \Omega^J)$ does not imply that $u \in L^1(\Theta_{\infty,0} \times \Omega^J)$, nor that $u \in Dom\Phi$.

Theorem: Let $F \in L^2(\Omega^J)$. Then $F \in DomD^J$ if and only if $\Psi F \in L^2(\Theta_{\infty,0} \times \Omega^J)$ and in this case we have

$$D^J F = \Psi F, \quad \nu \otimes \mathbb{P} - \text{a.e.}$$

Intrinsic operators and multiple stochastic integrals IX

Proof: Note that $F \in Dom\Psi$ because $Dom\Psi$ is the entire $L^0(\Omega^J)$.

Consider $u_\theta = I_k(g_k(\cdot,\theta))$ and assume $u \in Dom\Phi$. Then, by the duality property we have formally that

$$\mathbb{E}\int_{\Theta_{\infty,0}} D^J_\theta F u_\theta \nu(d\theta) = \mathbb{E}(F\delta^J(u)) = \mathbb{E}(F\Phi(u)) = \mathbb{E}\int_{\Theta_{\infty,0}} \Psi_\theta F u_\theta \nu(d\theta). \quad (7)$$

Intrinsic operators and multiple stochastic integrals X

The objects in (7) are well defined either if $F \in DomD^J$ or if $\Psi F \in L^2(\Theta_{\infty,0} \times \Omega^J)$.

In particular the previous equalities are true in the case

$$g_k(\theta_1, \ldots, \theta_k, \theta) := \mathbf{1}_{A_1}(\theta_1) \cdots \mathbf{1}_{A_k}(\theta_k) \mathbf{1}_A(\theta),$$

for any collection of pairwise disjoint and measurable sets $A_1, \ldots, A_k, A$ with finite measure ν.

In fact in this case $u \in L^1(\Theta_{\infty,0} \times \Omega^J) \subseteq Dom\Phi$. So, in particular we have

$$\mathbb{E}(I_k(\mathbf{1}^{\otimes k}_{A_1 \times \cdots \times A_k}) \int_A D^J_\theta F \nu(d\theta)) = \mathbb{E}(I_k(\mathbf{1}^{\otimes k}_{A_1 \times \cdots \times A_k}) \int_A \Psi_\theta F \nu(d\theta)).$$

By linearity and continuity we conclude that $\Psi F = D^J F$, $\nu \otimes \mathbb{P} - a.e.$

Intrinsic operators and multiple stochastic integrals XI

Theorem: Let $u \in L^2(\Theta_{\infty,0} \times \Omega^J) \cap Dom\Phi$. Then $u \in Dom\delta^J$ if and only if $\Phi u \in L^2(\Omega^J)$ and in this case we have

$$\delta^J u = \Phi u, \quad \mathbb{P} - \text{a.s.}$$

Proof: Let $G = I_k(g_k)$. Note that G is in $DomD^J$. Then, from previous results we have formally that

$$\mathbb{E}(\delta^J(u)G) = E\int_{\Theta_{\infty,0}} u_\theta D^J_\theta G\,\nu(d\theta) = \mathbb{E}\int_{\Theta_{\infty,0}} u_\theta \Psi_\theta G\,\nu(d\theta) = E(G\Phi(u)). \tag{8}$$

The objects in (8) are well defined if either $\Phi(u) \in L^2(\Omega^J)$ or if $u \in Dom\delta^J$ hold. Then the conclusion follows.

Remark: Similar results can be obtained for the operators $\bar{\Phi}$ and $\bar{\Psi}$.

The CHO formula in the pure jump case I

As an application of the previous results in the pure jump case we present a CHO type formula as an integral representation of random variables in $L^1(\Omega^J)$.

This in particular extends the formula proved in Picard (1996) for the standard Poisson case, as well as the formulae of CHO type proved in the L^2 setting, see for example Di Nunno-Øksendal-Proske (1999).

Theorem: Let $F \in L^1(\Omega^J)$ and assume $\Psi F \in L^1(\Theta_{\infty,0} \times \Omega^J)$. Then

$$F = \mathbb{E}(F) + \Phi(\mathbb{E}(\Psi_{t,x} F | \mathcal{F}_{t-})) \qquad \mathbb{P} - a.s.$$

Proof

The argument is organized in two steps.

In the first step we assume first that we are in Ω_m^J. In this case, ν is a finite measure concentrated on Θ_m.

Given $F \in L^1(\Omega^J)$ we can define, for every $n \geq 1$, F_n such that $F_n = F$ if $|F| \leq n$, $F_n = n$ if $F_n \geq n$ and $F_n = -n$ if $F \leq -n$. Of course, $F_n \in L^2(\Omega^J)$. And moreover $|F_n| \leq |F|$, for any n and

$$|\Psi F_n| \leq |TF_n| + |F_n| \leq |TF| + |F| \leq |\Psi F| + 2|F|.$$

Applying the CHO formula for square integrable functionals and previous results we obtain

$$F_n = \mathbb{E}(F_n) + \Phi(\mathbb{E}(\Psi_\theta F_n | \mathcal{F}_{t-})), \quad \mathbb{P} - a.s.$$

The CHO formula in the pure jump case III

Being ν finite, we note that $\Psi F_n \in L^2(\Theta_{\infty,0} \times \Omega^J)$ and $E(\Psi_\theta F_n|\mathcal{F}_{t-}) \in L^2(\Theta_{\infty,0} \times \Omega^J) \cap Dom\Phi$.

Using $E(\Psi_\theta F_n|\mathcal{F}_{t-})$ is predictable, we obtain

$$F_n = \mathbb{E}(F_n) + \int_{\Theta_m} \mathbb{E}(\Psi_\theta F_n|\mathcal{F}_{t-})\tilde{N}(d\theta), \quad \mathbb{P} - a.s.$$

Clearly, $F_n - \mathbb{E}(F_n)$ converges in L^1 to $F - \mathbb{E}(F)$. So, to prove the formula, for $F \in L^1(\Omega_m^J)$, it is enough to prove that

$$\int_{\Theta_m} \mathbb{E}(\Psi_\theta(F - F_n)|\mathcal{F}_{t-})\tilde{N}(d\theta) \longrightarrow_{n\uparrow\infty} 0,$$

with convergence in $L^1(\Omega^J)$.

The CHO formula in the pure jump case IV

Indeed we have

$$|\int_{\Theta_m} \mathbb{E}(\Psi_\theta(F-F_n)|\mathcal{F}_{t-})\tilde{N}(d\theta)| \leq \int_{\Theta_m} \mathbb{E}(|\Psi_\theta(F-F_n)||\mathcal{F}_{t-})N(d\theta)+\int_{\Theta_m} \mathbb{E}(|\Psi_\theta(F-F_n)||\mathcal{F}_{t-})\nu(d$$

So, it is enough to show that both terms of the sum on the right-hand side converge to 0. Observe that these two quantities have the same expectation, which is equal to

$$\mathbb{E}\int_{\Theta_m} |\Psi_\theta(F-F_n)|\nu(d\theta).$$

Now, the sequence $|\Psi(F-F_n)|$ converges to 0, $\mathbb{P}$-a.s., and it is dominated by

$$|\Psi(F-F_n)| \leq |\Psi F| + |\Psi F_n| \leq 2(|\Psi F| + |F|),$$

as this last quantity belongs to $L^1(\Theta_m \times \Omega_m^J)$ by hypothesis.

The CHO formula in the pure jump case V

Now we consider the general case. Then we have

$$F\mathbf{1}_{\Omega_m^J} - \mathbb{E}(F\mathbf{1}_{\Omega_m^J}) = \mathbf{1}_{\Omega_m^J}\int_{\Theta_m} \mathbb{E}(\Psi_\theta F|\mathcal{F}_{t-})\tilde{N}(d\theta), \quad \mathbb{P}-a.s.$$

It is immediate to see that if $F \in L^1(\Omega^J)$ the left-hand side of the equality converges to $F - \mathbb{E}(F)$. The convergence of the right-hand side is a consequence of the fact that

$$\mathbb{E}\left(\mathbf{1}_{\Omega_m^J}\left|\int_{\Theta_m} \mathbb{E}(\Psi_\theta F|\mathcal{F}_{t-})\tilde{N}(d\theta)\right|\right) \leq 2\int_{\Theta_{\infty,0}} \mathbb{E}(|\Psi_\theta F|)\nu(d\theta)$$

and the dominated convergence.

The CHO formula in the pure jump case VI

Observe that under the conditions of the previous theorem we have

$$\Psi_{s,x}\mathbb{E}[F|\mathcal{F}_{t-}] = \mathbb{E}[\Psi_{s,x}F|\mathcal{F}_{t-}]\mathbf{1}_{[0,t)}, \quad \nu\otimes\mathbb{P}-a.e.$$

Indeed, on Ω_m^J we consider the functionals F_n introduced in the proof of the previous theorem and we have

$$\Psi_{s,x}\mathbb{E}[F_n|\mathcal{F}_{t-}] = \mathbb{E}[\Psi_{s,x}F_n|\mathcal{F}_{t-}]\mathbf{1}_{[0,t)}, \quad \nu\otimes\mathbb{P}-a.e.$$

The sequence $\Psi_{s,x}F_n$ converges a.s. to $\Psi_{s,x}F$ and the term is bounded in $L^1(\Theta_{\infty,0}\times\Omega_m^J)$, so the right-hand side term converges to $\mathbb{E}[\Psi_{s,x}F|\mathcal{F}_{t-}]\mathbf{1}_{[0,t)}\mathbf{1}_{\Omega_m^J}$. Then, the left-hand side has a limit in L^1. On other hand, this left-hand side term also converges $\nu\otimes\mathbb{P}$-a.e. to $\Psi_{s,x}\mathbb{E}[F|\mathcal{F}_{t-}]$. So, the result follows.

Example 1 I

Consider a pure jump additive process L, i.e. for all t, L_t can be represented by the following Lévy-Itô decomposition:

$$L_t = \Gamma_t + \int_0^t \int_{\{|x|>1\}} xN(ds, dx) + \int_0^t \int_{\{|x|\leq 1\}} x\tilde{N}(ds, dx).$$

Consider L_T (for $T > 0$). If we assume $\mathbb{E}(|L_T|) < \infty$, or equivalently that

$$\int_0^T \int_{\{|x|>1\}} |x|\nu(ds, dx) < \infty$$

then we can write

$$L_T = \Gamma_T + \int_0^T \int_{\{|x|>1\}} x\nu(ds, dx) + \int_0^T \int_{\mathbb{R}} x\tilde{N}(ds, dx).$$

Example 1 II

On the other hand, applying the CHO formula, we have

$$\Psi_{s,x} L_T = x \mathbf{1}_{[0,T]}(s)$$

and

$$\mathbb{E}(\Psi_{s,x} L_T | \mathcal{F}_{s-}) = x \mathbf{1}_{[0,T)}(s).$$

So, the conditions to apply the CHO formula are equivalent to

$$\mathbb{E} \int_0^T \int_{\mathbb{R}} |x| \nu(ds, dx) < \infty. \tag{9}$$

So, under this condition, the CHO formula gives

$$L_T = \mathbb{E}(L_T) + \int_0^T \int_{\mathbb{R}} x \tilde{N}(ds, dx).$$

Example 1 III

This is clearly coherent with the Lévy-Itô decomposition because, under (9), we have

$$\mathbb{E}(L_T) = \Gamma_T + \int_0^T \int_{\{|x|>1\}} x\nu(ds, dx).$$

Example 2 I

Let $X := \{X_t, t \in [0,T]\}$ be a pure jump Lévy process with triplet $(\gamma_L t, 0, \nu_L t)$. Let $S_t := e^{X_t}$ be an asset price process. See for example Cont and Tankov (2004) for the use of exponential Lévy models in finance. Let $\mathbb{Q}$ be a risk-neutral measure. Recall that $e^{-rt}e^{X_t}$ is a $\mathbb{Q}-$martingale under the following assumptions on ν_L and γ_L:

$$\int_{|x|\geq 1} e^x \nu_L(dx) < \infty$$

and

$$\gamma_L = \int_{\mathbb{R}} (e^y - 1 - y\mathbf{1}_{\{|y|<1\}})\nu(dy).$$

See Cont and Tankov (2004) or Jafari-Vives (2013) for details.

Example 2 II

These conditions allow us to write without lost of generality,

$$X_t = x + (r - c_2)t + \int_0^t \int_{\mathbb{R}} y\tilde{N}(ds, dy),$$

where

$$c_2 := \int_{\mathbb{R}} (e^y - 1 - y)\nu_L(dy)$$

and N is a Poisson random measure under $\mathbb{Q}$.

According to CHO formula, if $F = S_T \in L^1(\Omega^J)$ and $\mathbb{E}_{\mathbb{Q}}[\Psi_{s,x} S_T | \mathcal{F}_{s-}] \in L^1([0, T] \times \Omega^J)$ we have

$$S_T = \mathbb{E}_{\mathbb{Q}}(S_T) + \int_{\Theta_{T,0}} \mathbb{E}_{\mathbb{Q}}[\Psi_{s,x} S_T | \mathcal{F}_{s-}]\tilde{N}(ds, dx).$$

Example 2 III

Observe that $\Psi_{s,x} S_T(\omega) = S_T(e^x - 1), \quad \ell \times \nu_L \times \mathbb{Q} - a.s.$, and this process belongs to $L^1(\Theta_{\infty,0} \times \Omega^J)$ if and only if $\int_{\mathbb{R}} |e^x - 1| \nu_L(dx) < \infty$. Here ℓ denotes the Lebesgue measure on $[0, T]$.

Then, in this case, we have

$$S_T = \mathbb{E}_{\mathbb{Q}}(S_T) + \int_{\Theta_{T,0}} e^{r(T-s)}(e^x - 1) S_{s-} \tilde{N}(ds, dx).$$

So, this result covers Lévy processes with finite activity and Lévy processes with infinite activity but finite variation.

The CHO formula in the Gaussian case I

Now we want to present a version of the CHO formula for the Gaussian case that extends the formula from the L^2 setting to the L^1 setting.

Recall that we have introduced $\mathcal{S}$ be the space of smooth functionals of type $F = f(W(h_1), \ldots, W(h_n))$ where f is a polynomial and $h_1, \ldots, h_n$ are elements of $L^2([0,\infty), \beta)$.

For a given $F \in \mathcal{S}$, its Malliavin derivative $\mathcal{D}$ can be defined as

$$\mathcal{D}^W F := \sum_{i=1}^{n} (\partial_i f)(W(h_1), \ldots, W(h_n)) h_i.$$

The CHO formula in the Gaussian case II

Associated to these definition and, for any $p \geq 1$, we can define the space $\mathbb{D}^{1,p}$ as the closure of $\mathcal{S}$ with respect the norm

$$||F||_{1,p} := (\mathbb{E}(|F|^p + ||\mathcal{D}^W F||_H^p)^{\frac{1}{p}}.$$

In particular we can consider the spaces $\mathbb{D}^{1,2}$ and $\mathbb{D}^{1,1}$, as the closures with respect the norms

$$||F||_{1,2} := (\mathbb{E}(|F|^2 + ||\mathcal{D}^W F||_H^2))^{\frac{1}{2}}.$$

and

$$||F||_{1,1} := E(|F|) + E(||\mathcal{D}^W F||_H),$$

respectively. Observe that we have the inclusions $\mathbb{D}^{1,p} \subseteq L^p(\Omega^W)$ and that $\mathbb{D}^{1,2} \subseteq \mathbb{D}^{1,1}$. By closure, the Malliavin derivative can be defined in any space $\mathbb{D}^{1,p}$.

The CHO formula in the Gaussian case III

For any $F \in \mathbb{D}^{1,1}$, we have the following version of the CHO formula (see Karatzas, Ocone and Li (1991)):

Theorem For any $T > 0$ and $F \in \mathbb{D}^{1,1}$, we have

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}(\mathcal{D}_t^W F | \mathcal{F}_{t-}) dW_t \qquad \mathbb{P} - a.s.$$

In this case we can also relate the operator $\mathcal{D}^W$ with the operator $D_{t,0}$, which is restricted to the Gaussian case. We have also the following results (Nualart (2006)):

Proposition Let $F \in L^2(\Omega^W)$ such that $F \in Dom\mathcal{D}^W$. Then $\mathcal{D}^W F \in L^2([0,\infty) \times \Omega^W)$ if and only if $F \in Dom D_{t,0}$ and in this case,

$$D_{t,0} F = \mathcal{D}_t^W F. \tag{10}$$

The CHO formula in additive case I

Now, for $F \in L^0(\Omega^W \times \Omega^J)$, we define the operator

$$\nabla_{t,x} F := \mathbf{1}_{\{0\}}(x)\mathcal{D}_t^W F + \mathbf{1}_{\mathbb{R}_0}(x)\Psi_{t,x} F \tag{11}$$

on the domain

$$Dom\nabla := \mathbb{D}^{1,1}(\Omega^W; L^0(\Omega^J)) \cap L^0(\Omega^J; L^0(\Omega^W)).$$

Note that ∇ extends $D_{t,x}$ from $\mathbb{D}^{1,2}(\Omega)$ to $Dom\nabla$. Note also that in the right-hand side of (11), if $\sigma \equiv 0$ only the second term remains and if $\nu \equiv 0$ only the first term remains.

The CHO formula in additive case II

Moreover, we have the following result

Corollary: If $F \in L^2(\Omega) \cap Dom\nabla$, we have

$$\Psi F \in L^2(\Theta_{\infty,0} \times \Omega) \text{ and } \mathcal{D}F \in L^2([0,\infty) \times \Omega) \Longleftrightarrow F \in DomD,$$

and in this case

$$D_{t,x}F = \nabla_{t,x}F \qquad \mu \times \mathbb{P} - a.e. \tag{12}$$

The CHO formula in additive case III

Hence we can extend the CHO formula to the following theorem:

Theorem Let $F \in L^1(\Omega) \cap Dom\nabla$ and assume $\Psi F \in L^1(\Theta_{\infty,0} \times \Omega)$. Then,

$$F = \mathbb{E}(F) + \int_{\Theta_{\infty,0}} \mathbb{E}(\Psi_{s,x} F | \mathcal{F}_{s-}) \tilde{N}(ds, dx) + \int_0^\infty \mathbb{E}(\mathcal{D}_s^W F | \mathcal{F}_{s-}) dW_s^\sigma \qquad \mathbb{P} - a.s.$$

Remark: This CHO formula identifies the kernels of the predictable representation property proved in Cohen (2013) in the case of additive integrators.

Application I

Assume X is a Lévy process with that ν finite. Let $F \in L^2(\Omega)$ such that $F \in DomD$ and

$$\int_{[0,T]\times\mathbb{R}} \left(D_\theta F\right)^2 \mu(d\theta) < \infty.$$

Then,

$$F = E[F] + \sigma \int_0^T \mathbb{E}\left[D_{t,0}F|\mathcal{F}_t^X\right] dW_t + \iint_{[0,T]\times\mathbb{R}_0} \mathbb{E}\left[D_{t,x}F|\mathcal{F}_{t-}^X\right] d\widetilde{N}(t,x),$$

Let be $m_1 = E(X_1)$. The process $\widetilde{X}_t = X_t - m_1 t$ is a square integrable martingale and every predictable process $g = \{g(t),\ t \in [0, T]\}$ with $E \int_0^T g(t)^2 dt < \infty$, will be $\widetilde{X}$ integrable and

$$\int_0^T g(t)\, d\widetilde{X}_t = \int_{[0,T]\times\mathbb{R}} g(t)\, dM(t, x).$$

The next result is a version of Galtchouk theorem and gives a representation of a random variable as an integral with respect to the process $\widetilde{X}$. For Lévy processes has been proved in Bent-Di Nunno-Løkka-Oksendal-Proske (2003).

Consider the measure $d\eta(x) = \sigma^2\, d\delta_0(x) + d\nu(x)$ and assume $F \in DomD$. Then

$$F = E[F] + \int_0^T h(t)\, d\widetilde{X}_t + N,$$

where

$$h(t) = \frac{\sigma^2\, \mathbb{E}\left[D_{t,0}F|\mathcal{F}_t^X\right] + \int_{\mathbb{R}} \mathbb{E}\left[D_{t,x}F|\mathcal{F}_{t-}^X\right] d\nu(x)}{\sigma^2 + \int_{\mathbb{R}} d\nu(x)}$$

and N is a square integrable random variable that is orthogonal to every integral $\int_0^T g(t) d\widetilde{X}_t$.

Quadratic hedging of Asiatic options I

Here we build the quadratic hedging of an Asiatic option in a market driven by a jump-diffusion process

$$X_t = \sigma W_t + \sum_{j=1}^{N_t} Z_j,$$

where the Z_j are i.i.d. with absolutely continuous distribution and $\mathbb{P}\{Z_j > -1\} = 1$, $\mathbb{E}[\exp\{2Z\}] < \infty$, and $\{N_t,\ t \geq 0\}$ is a Poisson process of parameter λ, independent of the Brownian motion $\{W_t,\ t \geq 0\}$. We will assume that the interest rate is 0.

Let $\{S_t,\ t \in [0,T]\}$ be an asset defined by $dS_t = S_{t-}\, dX_t$, that is

$$S_t = S_0 \exp\{\sigma W_t - \sigma^2 t/2\} \prod_{j=1}^{N_t} (1 + Z_j).$$

Quadratic hedging of Asiatic options II

Consider an Asiatic option given by the functional

$$F = \Big(\frac{1}{T}\int_0^T S_u\,du - K\Big)^+.$$

In agreement with Galtchouk theorem, the quadratic hedging is given by

$$h(t) = \frac{\sigma^2\,\mathbb{E}\big[D_{t,0}F|\mathcal{F}_t^X\big] + \int_{\mathbb{R}} \mathbb{E}\big[D_{t,x}F|\mathcal{F}_{t-}^X\big]\,\nu(dx)}{\sigma^2 + \int_{\mathbb{R}} \nu(dx)}.$$

Computation of jump derivatives I

First we compute the derivatives with respect (t, x), $x \neq 0$:

$$D_{t,x}F = \Bigg[\Big(\underbrace{\frac{1}{T}\int_0^t S_u\, du + (1+x)\frac{1}{T}\int_t^T S_u\, du - K}_{(1)} \Big)^+ - \Big(\underbrace{\frac{1}{T}\int_0^T S_u\, du - K}_{(2)} \Big)^+ \Bigg].$$

There are four cases:

Computation of jump derivatives II

Case 1. $(1) \geq 0$ and $(2) \geq 0$. Then

$$D_{t,x}F = \frac{x}{T}\int_t^T S_u\, du.$$

Case 2. $(1) \geq 0$ and $(2) < 0$. Then

$$D_{t,x}F = \Big(\frac{1}{T}\int_0^T S_u\, du + \frac{x}{T}\int_t^T S_u\, du - K\Big).$$

Case 3. $(1) < 0$ and $(2) \geq 0$. Then

$$D_{t,x}F = -\Big(\frac{1}{T}\int_0^T S_u\, du - K\Big).$$

Case 4. $(1) < 0$ and $(2) < 0$. Then $D_{t,x}F = 0$.

Computation of jump derivatives III

It follows that we need to compute a expression like

$$\mathbb{E}\Big[f(x, \int_0^t S_u\, du, \int_t^T S_u\, du)|\mathcal{F}_{t-}\Big],$$

for a certain function f.

Note that

$$\int_t^T S_u\, du = S_{t-} \int_t^T \frac{S_u}{S_{t-}}\, du$$

and

$$\int_t^T \frac{S_u}{S_{t-}}\, du$$

is independent of $\mathcal{F}_{t-}$.

Computation of jump derivatives IV

Then

$$\mathbb{E}\Big[f\big(x, \textstyle\int_0^t S_u\,du, \int_t^T S_u\,du\big)|\mathcal{F}_{t-}\Big] = \mathbb{E}\Big[f\big(x, \alpha, \beta \textstyle\int_t^T \frac{S_u}{S_{t-}}\,du\big)\Big]\Big|_{\alpha=\int_0^t S_u\,du,\, \beta=S_{t-}}.$$

Moreover, we have the equality in law

$$\mathcal{L}\Big(\int_t^T \frac{S_u}{S_{t-}}\,du\Big) = \mathcal{L}\Big(\frac{1}{S_0}\int_0^{T-t} S_u\,du,\Big).$$

Computation of jump derivative V

Conditionally to $N_\tau = k$,

$$\frac{1}{S_0}\int_0^\tau S_u\,du$$

$$= \sum_{j=0}^{k-1}\int_{T_j}^{T_{j+1}} \exp\{\sigma W_u - \sigma^2 u/2\}\,du \prod_{i=0}^{j}(1+Z_i) + \int_{T_k}^{\tau} \exp\{\sigma W_u - \sigma^2 u/2\}\,du \prod_{i=0}^{k}(1+Z_i),$$

where $T_0 = 0$ and $Z_0 = 0$. The law of $(T_1, \ldots, T_k)$ conditionally to $N_\tau = k$ is a k-dimensional Dirichlet on $(0, \tau)$.

Therefore, the expectation $\mathbb{E}\left[f\left(x, \alpha, \beta \int_0^t \frac{S_u}{S_{r-}}\,du\right)\right]$ can be computed, and

$$\mathbb{E}\left[f(x, \int_0^t S_u\,du, \int_t^T S_u\,du)|\mathcal{F}_{t-}\right].$$

Computation of Brownian derivative I

Now we deal with the derivative of F with respect to the Brownian motion. Denote by Φ_n the distribution function of a normal random variable with variance $1/n$, and let

$$g_n(x) = \int_{-\infty}^{x} \Phi_n(t)\, dt \in \mathcal{C}^\infty.$$

We have that $\lim_n g_n(x) \to x^+$ uniformly on $x \in \mathbb{R}$.

On the other hand, $g_n(\frac{1}{T}\int_0^T S_u\, du - K) \in \mathrm{Dom} D^0$.

It follows that $(\frac{1}{T}\int_0^T S_u\, du - K)^+ \in DomD^0$ and

$$\lim_n D_{t,0} g_n(\frac{1}{T}\int_0^T S_u\, du - K) = D_{t,0}(\frac{1}{T}\int_0^T S_u\, du - K)^+$$

in the weak topology of $L^2([0,T] \times \Omega)$.

Computation of Brownian derivative I

However, for each t, a.s.,

$$\lim_n D_{t,0} g_n\left(\frac{1}{T}\int_0^T S_u\,du - K\right) = \mathbf{1}_{\{\frac{1}{T}\int_0^T S_u\,du > K\}} \frac{1}{T}\int_0^T D_{t,0} S_u\,du$$

$$= \mathbf{1}_{\{\frac{1}{T}\int_0^T S_u\,du > K\}} \frac{1}{T}\int_t^T S_u\,du,$$

and we deduce

$$D_{t,0}\left(\frac{1}{T}\int_0^T S_u\,du - K\right)^+ = \mathbf{1}_{\{\frac{1}{T}\int_t^T S_u\,du > K\}} \frac{1}{T}\int_t^T S_u\,du.$$

Finally, the expectation $\mathbb{E}[D_{t,0}F|\mathcal{F}_t]$ can be computed following the same rules as before.