Fractional Vasicek model: Asymptotic properties and statistical inference

Kostiantyn Ralchenko

University of Vaasa
Taras Shevchenko National University of Kyiv

42nd Finnish Summer School on Probability and Statistics
May 27–31, 2024
Lammi, Finland

Lecture 1

Introduction

We deal with the fractional Vasicek model of the form

$$dX_t = (\alpha - \beta X_t)\,dt + \gamma dB_t^H, \quad X_0 = x_0 \in \mathbb{R}, \tag{1}$$

where B^H is a fractional Brownian motion with Hurst index $H \in (0,1)$.
It is a generalization to the fractional case of the classical interest rate model proposed by O. Vasicek in 1977. This generalization enables to study processes with long-range dependence, which arise in financial mathematics and several other areas such as telecommunication networks, investigation of turbulence and image processing. In recent years, many articles on various financial applications of the fractional Vasicek model (1) have appeared.
In order to use this model in practice, a theory of parameter estimation is necessary.

Financial applications

A. Chronopoulou and F. G. Viens.
Estimation and pricing under long-memory stochastic volatility.
Ann. Finance, 8(2-3):379–403, 2012.

Sylvain Corlay, Joachim Lebovits, and Jacques Lévy Véhel.
Multifractional stochastic volatility models.
Math. Finance, 24(2):364–402, 2014.

H. Fink, C. Klüppelberg, and M. Zähle.
Conditional distributions of processes related to fractional Brownian motion.
J. Appl. Probab., 50(1):166–183, 2013.

R. Hao, Y. Liu, and S. Wang.
Pricing credit default swap under fractional Vasicek interest rate model.
Journal of Mathematical Finance, 4(1):10–20, 2014.

Lina Song and Kele Li.
Pricing option with stochastic interest rates and transaction costs in fractional Brownian markets.
Discrete Dyn. Nat. Soc., pages Art. ID 7056734, 8, 2018.

Weilin Xiao, Weiguo Zhang, Xili Zhang, and Xiaoyan Chen.
The valuation of equity warrants under the fractional Vasicek process of the short-term interest rate.
Phys. A, 394:320–337, 2014.

Relation to fractional Ornstein–Uhlenbeck process

- Fractional Ornstein–Uhlenbeck process:
  - Equation:
$$dX_t = x_0 + \theta X_t\,dt + \sigma\,dB_t^H, \quad t \ge 0,$$
  - Solution:
$$X_t = x_0 e^{\theta t} + \sigma \int_0^t e^{\theta(t-s)}\,dB_s^H.$$
- Fractional Vasicek model:
  - Equation:
$$dX_t = (\alpha - \beta X_t)\,dt + \gamma\,dB_t^H, \quad t \ge 0,$$
  - Solution:
$$X_t = \underbrace{x_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)}_{\text{deterministic function}} + \underbrace{\gamma \int_0^t e^{-\beta(t-s)}\,dB_s^H}_{\text{fO-U process}}.$$

Fractional Brownian motion

Let $(\Omega, \mathfrak{F}, \mathrm{P})$ be a complete probability space.

Definition 1.1

Fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \left\{B_t^H,\ t \in \mathbb{R}^+\right\}$ on $(\Omega, \mathfrak{F}, \mathrm{P})$ featuring the properties

(i) $B_0^H = 0$;

(ii) $\mathrm{E}B_t^H = 0,\ t \in \mathbb{R}^+$;

(iii) $\mathrm{E}B_t^H B_s^H = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),\ s, t \in \mathbb{R}^+$.

Remark 1.2

From time to time, especially for introducing a *stationary fractional Ornstein–Uhlenbeck process*, we will need also the fBm defined on the whole $\mathbb{R}$. The difference is that in this case $\mathrm{E}B_t^H B_s^H = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),\ s, t \in \mathbb{R}$.

Basic properties of fBm

The following statements can be derived directly from the above definition.

1. If $H = \frac{1}{2}$, then a fractional Brownian motion is a standard Wiener process.
2. A fractional Brownian motion is *self-similar* with the self-similarity parameter H, that is, $\{B^H_{ct}\} \overset{d}{=} \{c^H B^H_t\}$ for any $c > 0$. Here $\overset{d}{=}$ means that all finite-dimensional distributions of both processes coincide.
3. A fractional Brownian motion has *stationary increments* that is implied by the form of its incremental covariance:

$$\mathrm{E}\left(B^H_t - B^H_s\right)^2 = (t-s)^{2H}. \tag{2}$$

4. The increments of a fractional Brownian motion are independent only in the case $H = 1/2$. They are negatively correlated for $H \in (0, 1/2)$ and positively correlated for $H \in (1/2, 1)$.

Further properties:

5. For $H \in (1/2, 1)$ an fBm has the property of *long-range dependence*. The aforementioned means that $\sum_{n=1}^{\infty} r(n) = \infty$, where $r(n) = \mathrm{E} B_1^H \left(B_{n+1}^H - B_n^H \right) > 0$ is the autocovariance function.
6. Due to the Kolmogorov continuity theorem, the property (2) implies that a fractional Brownian motion has a continuous modification. Moreover, this modification is γ-*Hölder continuous* on each finite interval for any $\gamma \in (0, H)$.
7. It is also well-known that a fractional Brownian motion is not a process of bounded variation.
8. If $H \neq \frac{1}{2}$, then it is neither a semimartingale nor a Markov process.
9. For any $p > 1$ and any $H \in (0, 1)$ there exists a nonnegative random variable $\xi = \xi(p, H)$ such that for all $t \geq 0$,

$$\sup_{0 \leq s \leq t} \left| B_s^H \right| \leq \left(\left(t^H (\log^+ t)^p \right) \vee 1 \right) \xi, \tag{3}$$

and there exists $c_\xi(p, H) > 0$ such that for any $0 < y < c_\xi(p, H)$, $\mathrm{E} \exp\{y \xi^2\} < \infty$.

Selected books on fBm

F. Biagini, Y. Hu, B. Øksendal, and T. Zhang.
Stochastic Calculus for Fractional Brownian Motion and Applications.
Springer, 2008.

Y. Mishura.
Stochastic Calculus for Fractional Brownian Motion and Related Processes,
volume 1929 of *Lecture Notes in Mathematics*.
Springer, 2008.

Y. Mishura and M. Zili.
Stochastic Analysis of Mixed Fractional Gaussian Processes.
ISTE and Elsevier, 2018.

I. Nourdin.
Selected Aspects of Fractional Brownian Motion,
volume 4 of *Bocconi & Springer Series*.
Springer, 2012.

The present subsection deals with the Langevin equation that will be written as

$$X_t = x_0 + \theta \int_0^t X_s \, ds + \sigma B_t^H, \quad t \geq 0, \tag{4}$$

where $x_0 \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\sigma > 0$ are real parameters, and B^H is an fBm with the Hurst index H. Since the model does not contain stochastic integrals with respect to fBm, it is possible to consider an fBm with arbitrary $H \in (0,1)$. For all $H \in (0,1)$ the equation (4) has a unique solution

$$X_t = x_0 e^{\theta t} + \theta \sigma e^{\theta t} \int_0^t e^{-\theta s} B_s^H \, ds + \sigma B_t^H, \quad t \geq 0, \tag{5}$$

which is called a fractional Ornstein–Uhlenbeck process (fO-U).

Note that for $H \in (1/2, 1)$ the process (5) can be written in the following form

$$X_t = x_0 e^{\theta t} + \sigma \int_0^t e^{\theta(t-s)} dB_s^H,$$

where $\int_0^t e^{\theta(t-s)} dB_s^H$ is a Wiener integral w.r.t. the fBm. Evidently, the integral $\int_0^t e^{\theta(t-s)} dB_s^H$ exists for all $H \in (0, 1)$, if we understand it via integration by parts:

$$\int_0^t e^{-\theta s} dB_s^H = \theta \int_0^t e^{-\theta s} B_s^H \, ds + e^{-\theta t} B_t^H.$$

Let us calculate the covariance function of the fO-U and describe the properties of a stationary Gaussian process related to the model.

Theorem 2.1

Let $H \in (0,1)$, $t \geq s \geq 0$. *Then the covariance function of the fO-U* (5) *is given by*

$$\begin{aligned} &\operatorname{cov}(X_t, X_s) \\ &= \frac{H\sigma^2}{2} \Bigg(-e^{\theta t - \theta s} \int_0^{t-s} e^{-\theta z} z^{2H-1} dz + e^{-\theta t + \theta s} \int_{t-s}^{t} e^{\theta z} z^{2H-1} dz \\ &\qquad - e^{\theta t + \theta s} \int_s^t e^{-\theta z} z^{2H-1} dz + e^{\theta t - \theta s} \int_0^s e^{\theta z} z^{2H-1} dz \\ &\qquad + 2e^{\theta t + \theta s} \int_0^t e^{-\theta z} z^{2H-1} dz \Bigg). \end{aligned} \tag{6}$$

Proof. Using (5) and the formula for covariance of an fBm, we can write

$$
\begin{aligned}
\operatorname{cov}(X_t, X_s) &= \mathrm{E}\left[\left(X_t - x_0 e^{\theta t}\right)\left(X_s - x_0 e^{\theta s}\right)\right] \\
&= \mathrm{E}\left[\left(\theta\sigma e^{\theta t}\int_0^t e^{-\theta u} B_u^H du + \sigma B_t^H\right)\left(\theta\sigma e^{\theta s}\int_0^s e^{-\theta v} B_v^H dv + \sigma B_s^H\right)\right] \\
&= \frac{\theta\sigma^2}{2} e^{\theta t}\int_0^t e^{-\theta u}\left(u^{2H} + s^{2H} - |u-s|^{2H}\right) du \\
&\quad + \frac{\theta\sigma^2}{2} e^{\theta s}\int_0^s e^{-\theta v}\left(v^{2H} + t^{2H} - |v-t|^{2H}\right) dv \\
&\quad + \frac{\sigma^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \\
&\quad + \frac{\theta^2\sigma^2}{2} e^{\theta t + \theta s}\int_0^t \int_0^s e^{-\theta u - \theta v}\left(u^{2H} + v^{2H} - |u-v|^{2H}\right) du\, dv \\
&= \frac{\sigma^2}{2}\sum_{n=1}^{10} I_n,
\end{aligned}
$$

where

$$I_1 = \theta e^{\theta t} \int_0^t e^{-\theta u} s^{2H} du, \quad I_2 = \theta e^{\theta t} \int_0^t e^{-\theta u} u^{2H} du,$$

$$I_3 = -\theta e^{\theta t} \int_0^t e^{-\theta u} |u - s|^{2H} du, \quad I_4 = \theta e^{\theta s} \int_0^s e^{-\theta v} t^{2H} dv,$$

$$I_5 = \theta e^{\theta s} \int_0^s e^{-\theta v} v^{2H} dv, \quad I_6 = -\theta e^{\theta s} \int_0^s e^{-\theta v} (t - v)^{2H} dv,$$

$$I_7 = t^{2H} + s^{2H} - (t - s)^{2H}, \quad I_8 = \theta^2 e^{\theta t + \theta s} \int_0^t e^{-\theta v} dv \int_0^s e^{-\theta u} u^{2H} du,$$

$$I_9 = \theta^2 e^{\theta t + \theta s} \int_0^s e^{-\theta u} du \int_0^t e^{-\theta v} v^{2H} dv,$$

$$I_{10} = -\theta^2 e^{\theta t + \theta s} \int_0^t \int_0^s e^{-\theta u - \theta v} |u - v|^{2H} \, du \, dv.$$

The first two integrals are equal to

$$I_1 = s^{2H} \left(e^{\theta t} - 1 \right) \quad \text{and}$$

$$I_2 = -e^{\theta t}\int_0^t u^{2H} de^{-\theta u} = -t^{2H} + 2He^{\theta t}\int_0^t e^{-\theta u} u^{2H-1} du.$$

By changes of variables and integration by parts, we obtain

$$\begin{aligned}
I_3 &= -\theta e^{\theta t}\int_0^s e^{-\theta u}(s-u)^{2H} du - \theta e^{\theta t}\int_s^t e^{-\theta u}(u-s)^{2H} du \\
&= -\theta e^{\theta t-\theta s}\int_0^s e^{\theta z} z^{2H} dz - \theta e^{\theta t-\theta s}\int_0^{t-s} e^{-\theta z} z^{2H} dz \\
&= -e^{\theta t-\theta s}\bigg(e^{\theta s}s^{2H} - 2H\int_0^s e^{\theta z} z^{2H-1} dz - e^{-\theta(t-s)}(t-s)^{2H} \\
&\quad + 2H\int_0^{t-s} e^{-\theta z} z^{2H-1} dz\bigg) \\
&= -e^{\theta t}s^{2H} + (t-s)^{2H} + 2He^{\theta t-\theta s}\int_0^s e^{\theta z} z^{2H-1} dz \\
&\quad - 2He^{\theta t-\theta s}\int_0^{t-s} e^{-\theta z} z^{2H-1} dz.
\end{aligned}$$

Similarly to I_1, I_2, I_3, we transform I_4, I_5, I_6:

$$I_4 = t^{2H}\left(e^{\theta s} - 1\right), \qquad I_5 = -s^{2H} + 2He^{\theta s}\int_0^s e^{-\theta v} v^{2H-1} dv,$$

$$I_6 = -\theta e^{\theta s - \theta t}\int_{t-s}^t e^{\theta z} z^{2H} dz = -e^{\theta s-\theta t}\int_{t-s}^t z^{2H} de^{\theta z}$$
$$= -e^{\theta s} t^{2H} + (t-s)^{2H} + 2He^{\theta s - \theta t}\int_{t-s}^t e^{\theta z} z^{2H-1} dz.$$

Further,

$$I_8 = e^{\theta t + \theta s}\left(e^{-\theta t} - 1\right)\int_0^s u^{2H} de^{-\theta u}$$
$$= (1 - e^{\theta t}) s^{2H} - 2He^{\theta s}(1 - e^{\theta t})\int_0^s e^{-\theta u} u^{2H-1} du,$$

and similarly

$$I_9 = (1 - e^{\theta s}) t^{2H} - 2He^{\theta t}(1 - e^{\theta s})\int_0^t e^{-\theta v} v^{2H-1} dv$$

Finally, we consider the term I_{10}. It can be represented as the sum of the following integrals:

$$\begin{aligned} I_{10} = -\theta^2 e^{\theta t+\theta s} \int_0^s \int_0^v e^{-\theta u-\theta v}(v-u)^{2H}\,du\,dv \\ - \theta^2 e^{\theta t+\theta s} \int_0^s \int_v^s e^{-\theta u-\theta v}(u-v)^{2H}\,du\,dv \\ - \theta^2 e^{\theta t+\theta s} \int_s^t \int_0^s e^{-\theta u-\theta v}(v-u)^{2H}\,du\,dv \\ = -2\theta^2 e^{\theta t+\theta s} \int_0^s \int_0^v e^{-\theta u-\theta v}(v-u)^{2H}\,du\,dv \\ - \theta^2 e^{\theta t+\theta s} \int_s^t \int_0^s e^{-\theta u-\theta v}(v-u)^{2H}\,du\,dv =: I_{10}' + I_{10}''. \end{aligned}$$

Using the change of variables $v - u = z$, the change of order of integration, and integration by parts, we get

$$\begin{aligned} I_{10}' &= -2\theta^2 e^{\theta t + \theta s} \int_0^s e^{-2\theta v} \int_0^v e^{\theta z} z^{2H} dz dv \\ &= -2\theta^2 e^{\theta t + \theta s} \int_0^s e^{\theta z} z^{2H} \int_z^s e^{-2\theta v} dv dz \\ &= -2\theta^2 e^{\theta t + \theta s} \int_0^s e^{\theta z} z^{2H} \frac{e^{-2\theta s} - e^{-2\theta z}}{-2\theta} dz \\ &= \theta e^{\theta t - \theta s} \left(\int_0^s e^{\theta z} z^{2H} dz - \int_0^s e^{-\theta u} u^{2H} du \right) \\ &= e^{\theta t} s^{2H} - 2H e^{\theta t - \theta s} \int_0^s e^{\theta z} z^{2H-1} dz + e^{\theta t} s^{2H} \\ &\quad - 2H e^{\theta t + \theta s} \int_0^s e^{-\theta u} u^{2H-1} du. \end{aligned}$$

In order to simplify I''_{10}, we need to consider two cases. If $t > 2s$, then

$$\begin{aligned}
I''_{10} &= -\theta^2 e^{\theta t+\theta s} \int_s^t \int_{v-s}^{v} e^{\theta z - 2\theta v} z^{2H} dz dv \\
&= -\theta^2 e^{\theta t+\theta s} \Bigg(\int_0^s \int_s^{z+s} e^{\theta z - 2\theta v} z^{2H} dv dz \\
&\quad + \int_s^{t-s} \int_z^{z+s} e^{\theta z - 2\theta v} z^{2H} dv dz + \int_{t-s}^t \int_z^t e^{\theta z - 2\theta v} z^{2H} dv dz \Bigg) \\
&= -\theta^2 e^{\theta t+\theta s} \Bigg(\int_0^s e^{\theta z} z^{2H} \, \frac{e^{-2\theta(z+s)} - e^{-2\theta s}}{-2\theta} \, dz \\
&\quad + \int_s^{t-s} e^{\theta z} z^{2H} \frac{e^{-2\theta(z+s)} - e^{-2\theta z}}{-2\theta} \, dz + \int_{t-s}^t e^{\theta z} z^{2H} \frac{e^{-2\theta t} - e^{-2\theta z}}{-2\theta} \, dz \Bigg) \\
&= \frac{\theta}{2} e^{\theta t - \theta s} \int_0^{t-s} e^{-\theta z} z^{2H} dz - \frac{\theta}{2} e^{\theta t + \theta s} \int_s^t e^{-\theta z} z^{2H} dz \\
&\quad - \frac{\theta}{2} e^{\theta t - \theta s} \int_0^s e^{\theta z} z^{2H} dz + \frac{\theta}{2} e^{\theta s - \theta t} \int_{t-s}^t e^{\theta z} z^{2H} dz.
\end{aligned}$$

Integrating by parts in each integral, we arrive at the following equality

$$
\begin{aligned}
I''_{10} = e^{\theta s} t^{2H} - e^{\theta t} s^{2H} - H e^{\theta s - \theta t} \int_{t-s}^{t} e^{\theta z} z^{2H-1} dz \\
+ H e^{\theta t - \theta s} \int_0^s e^{\theta z} z^{2H-1} dz - H e^{\theta t + \theta s} \int_s^t e^{-\theta z} z^{2H-1} dz \\
+ H e^{\theta t - \theta s} \int_0^{t-s} e^{-\theta z} z^{2H-1} dz - (t-s)^{2H}.
\end{aligned}
$$

Similarly one can verify the last formula for the case $s < t < 2s$.
Thus, summing up all the terms, we get (6). $\square$

Corollary 2.2

The r. v. X_t has Gaussian distribution $\mathcal{N}\left(x_0 e^{\theta t}, v(\theta, t)\right)$, with variance

$$v(\theta, t) := \operatorname{Var} X_t = H\sigma^2 \int_0^t z^{2H-1} \left(e^{\theta z} + e^{\theta(2t-z)}\right) dz. \tag{7}$$

Corollary 2.2

The r. v. X_t has Gaussian distribution $\mathcal{N}\left(x_0 e^{\theta t}, v(\theta, t)\right)$, with variance

$$v(\theta, t) := \operatorname{Var} X_t = H\sigma^2 \int_0^t z^{2H-1} \left(e^{\theta z} + e^{\theta(2t-z)}\right) dz. \tag{7}$$

Let us investigate the asymptotic behavior of the variance $v(\theta, t)$ as $t \to \infty$.

Lemma 2.3

(*i*) *If* $\theta > 0$, *then* $v(\theta, t) \sim \frac{\sigma^2 H \Gamma(2H)}{\theta^{2H}} e^{2\theta t}$, *as* $t \to \infty$.

(*ii*) *If* $\theta < 0$, *then* $v(\theta, t) \to \frac{\sigma^2 H \Gamma(2H)}{(-\theta)^{2H}}$, *as* $t \to \infty$.

(*iii*) $v(0, t) = \sigma^2 t^{2H}$, $t \geq 0$.

Proof. (i) If $\theta > 0$, then by formula (7),

$$\frac{v(\theta,t)}{e^{2\theta t}} = H\sigma^2 e^{-2\theta t}\int_0^t z^{2H-1}e^{\theta z}dz + H\sigma^2\int_0^t z^{2H-1}e^{-\theta z}dz \to \frac{\sigma^2 H\Gamma(2H)}{\theta^{2H}},$$

as $t \to \infty$.
(ii) Note that $v(\theta,t) = e^{2\theta t}v(-\theta,t)$, by (7). Then the convergence follows from (i).
(iii) The statement follows directly from (7). $\square$

Stationary fractional Ornstein–Uhlenbeck process

Let $\theta < 0$. Consider an fBm defined on $\mathbb{R}$. In this case the process

$$Y_t = \sigma \int_{-\infty}^{t} e^{\theta(t-s)}\, dB_s^H := \theta\sigma e^{\theta t} \int_{-\infty}^{t} e^{-\theta s} B_s^H\, ds + \sigma B_t^H, \quad t \ge 0. \tag{8}$$

is well-defined.
Then the fO-U (5) can be expressed as

$$X_t = x_0 e^{\theta t} + Y_t - e^{\theta t} Y_0, \quad t \ge 0. \tag{9}$$

In view of the next result, the process Y is called a stationary fractional Ornstein–Uhlenbeck process.

Theorem 2.4

Let $\theta < 0$, $H \in (0,1)$. Then the centered Gaussian process $Y = \{Y_t, t \ge 0\}$ defined by (8) is stationary and ergodic.

The stationarity of Y follows immediately from the stationarity of the increments of an fBm. However, we can compute the covariance function explicitly.

Lemma 2.5

Let $\theta < 0$, $H \in (0,1)$. For $t \ge s \ge 0$, the covariance function of Y is given by

$$\operatorname{cov}(Y_t, Y_s) = \frac{H\sigma^2}{2}\Bigg(\frac{\Gamma(2H)}{(-\theta)^{2H}} e^{\theta(t-s)} - e^{\theta(t-s)} \int_0^{t-s} e^{-\theta z} z^{2H-1} dz$$
$$+ e^{-\theta(t-s)} \int_{t-s}^{+\infty} e^{\theta z} z^{2H-1} dz\Bigg) =: r(t-s).$$

Proof. By (9),

$$\operatorname{cov}(Y_t, Y_s) = \operatorname{cov}(X_t, X_s) + e^{\theta s}\operatorname{cov}(X_t, Y_0) + e^{\theta t}\operatorname{cov}(X_s, Y_0) + e^{\theta t+\theta s}\mathrm{E}\,Y_0^2. \tag{10}$$

The first term was calculated in Theorem 2.1. Similarly one can compute

$$\begin{aligned}
\operatorname{cov}(X_t, Y_0) &= \frac{H\sigma^2}{2}\left(-\frac{\Gamma(2H)}{(-\theta)^{2H}}e^{\theta t} - e^{\theta t}\int_0^t e^{-\theta z} z^{2H-1}dz\right.\\
&\qquad\qquad\left. + e^{-\theta t}\int_t^\infty e^{\theta z} z^{2H-1}dz\right),\\
\operatorname{cov}(X_s, Y_0) &= \frac{H\sigma^2}{2}\left(-\frac{\Gamma(2H)}{(-\theta)^{2H}}e^{\theta s} - e^{\theta s}\int_0^s e^{-\theta z} z^{2H-1}dz\right.\\
&\qquad\qquad\left. + e^{-\theta s}\int_s^\infty e^{\theta z} z^{2H-1}dz\right),\\
\mathrm{E}\,Y_0^2 &= \frac{\sigma^2 H\Gamma(2H)}{(-\theta)^{2H}}.
\end{aligned} \tag{11}$$

The proof can be concluded now by inserting these expressions together with (6) into (10) and using $\int_0^\infty e^{\theta z} z^{2H-1}dz = \frac{\Gamma(2H)}{(-\theta)^{2H}}$. $\square$

Remark 2.6

The second moment of Y_0 can be also calculated by applying Corollary 2.2 and Lemma 2.3. Indeed,

$$
\begin{aligned}
\mathrm{E}Y_0^2 &= \sigma^2 \lim_{t\to-\infty} \mathrm{E}\left(-e^{-\theta t}B_t^H + \theta\int_t^0 e^{-\theta s}B_s^H\,ds\right)^2 \\
&= \sigma^2 \lim_{t\to\infty} \mathrm{E}\left(-e^{\theta t}B_{-t}^H + \theta\int_0^t e^{\theta s}B_{-s}^H\,ds\right)^2 \\
&= \sigma^2 \lim_{t\to\infty} e^{2\theta t}\left(\mathrm{E}\left(B_{-t}^H\right)^2 - 2\theta e^{-\theta t}\int_0^t e^{\theta s}\mathrm{E}B_{-t}^H B_{-s}^H\,ds\right. \\
&\quad \left. + \theta^2 e^{-2\theta t}\int_0^t\int_0^t e^{\theta(s+u)}\mathrm{E}B_{-s}^H B_{-u}^H\,ds\,du\right) \\
&= \sigma^2 \lim_{t\to\infty} e^{2\theta t} v(-\theta, t) = \frac{\sigma^2 H\Gamma(2H)}{(-\theta)^{2H}}.
\end{aligned}
$$

In order to prove the ergodicity of the Gaussian stationary process Y it suffices to show that its autocovariance function $r(t) = \operatorname{cov}(Y_t, Y_0)$ vanishes at infinity.

Lemma 2.7

Let $\theta < 0$, $H \in (0,1)$. Then $r(t) \to 0$, as $t \to \infty$.

Proof. By the change of variables, we get

$$r(t) = \frac{H\sigma^2}{2(-\theta)^{2H}} \left(\Gamma(2H) e^{\theta t} - e^{\theta t} \int_0^{-\theta t} e^y y^{2H-1} dy + e^{-\theta t} \int_{-\theta t}^{+\infty} e^{-y} y^{2H-1} dy \right) \quad (12)$$

If $H = 1/2$, then $r(t) = -\frac{\sigma^2}{2\theta} e^{\theta t} \to 0$, as $t \to \infty$.

If $H \neq 1/2$, then using integration by parts, we obtain

$$r(t) = \frac{H\sigma^2}{2(-\theta)^{2H}} \left(-e^{\theta t} \int_1^{-\theta t} e^y y^{2H-1} dy + e^{-\theta t} \int_{-\theta t}^{+\infty} e^{-y} y^{2H-1} dy \right) + O\left(e^{\theta t}\right)$$

$$= \frac{H(2H-1)\sigma^2}{2(-\theta)^{2H}} \left(e^{\theta t} \int_1^{-\theta t} e^y y^{2H-2} dy + e^{-\theta t} \int_{-\theta t}^{+\infty} e^{-y} y^{2H-2} dy \right) + O\left(e^{\theta t}\right), \tag{13}$$

as $t \to \infty$. We have

$$e^{-\theta t} \int_{-\theta t}^{+\infty} e^{-y} y^{2H-2}\, dy \leq e^{-\theta t} \int_{-\theta t}^{+\infty} e^{-y} (-\theta t)^{2H-2}\, dy = (-\theta t)^{2H-2}, \tag{14}$$

and

$$e^{\theta t}\int_1^{-\theta t} e^y y^{2H-2}\,dy \le e^{\theta t}\left(\int_1^{-\frac{\theta t}{2}} e^y\,dy + \int_{-\frac{\theta t}{2}}^{-\theta t} e^y \left(-\frac{\theta t}{2}\right)^{2H-2} dy\right)$$
$$\le e^{\frac{\theta t}{2}} + \left(-\frac{\theta t}{2}\right)^{2H-2}. \tag{15}$$

Thus, both terms in (13) converge to zero as $t \to \infty$. This concludes the proof. □

Remark 2.8

The stationary fO-U was introduced and investigated in [Cheridito et al. (2003)]. Moreover, the following asymptotic relation for the autocovariance function $r(t)$ was established for $H \neq \frac{1}{2}$ and $N = 1, 2 \dots$:

$$r(t) = \frac{\sigma^2}{2} \sum_{n=1}^{N} (-\theta)^{-2n} \left(\prod_{k=0}^{2n-1} (2H - k) \right) t^{2H-2n} + O\left(t^{2H-2N-2}\right), \quad (16)$$

as $t \to \infty$. This implies that for $H \in (\frac{1}{2}, 1]$, the process Y exhibits long-range dependence, that is, $\sum_{n=0}^{\infty} r(n) = \infty$. The expansion (16) can be deduced from (13). In order to obtain this result, one should use integration by parts N times, and then bound the remaining terms similarly to (14)–(15).

Patrick Cheridito, Hideyuki Kawaguchi, and Makoto Maejima.
Fractional Ornstein-Uhlenbeck processes.
Electron. J. Probab., 8:1–14, 2003.

A similar relation holds for the fO-U X, namely for $H \neq \frac{1}{2}$, $N = 1, 2 \dots,$ and $s \geq 0$

$$
\begin{aligned}
&\operatorname{cov}(X_s, X_{s+t}) \\
&\quad = \frac{\sigma^2}{2} \sum_{n=1}^{N} (-\theta)^{-2n} \left(\prod_{k=0}^{2n-1} (2H - k) \right) \left(t^{2H-2n} - e^{\theta s} (s+t)^{2H-2n} \right) \\
&\qquad + O\left(t^{2H-2N-2} \right), \quad \text{as } t \to \infty.
\end{aligned}
$$

It follows immediately from (16) if we note that, by (9),

$$
\begin{aligned}
\operatorname{cov}(X_s, X_{s+t}) &= \operatorname{cov}(Y_s, Y_{s+t}) - e^{\theta s} \operatorname{cov}(Y_0, Y_{s+t}) \\
&\qquad - e^{\theta(s+t)} \operatorname{cov}(Y_s, Y_0) + e^{\theta(2s+t)} \mathrm{E} Y_0^2 \\
&= r(t) - e^{\theta s} r(s+t) + O\left(e^{\theta t} \right), \quad t \to \infty.
\end{aligned}
$$

Basic reference Lecture 1

Kęstutis Kubilius, Yuliya Mishura, and Kostiantyn Ralchenko.
Parameter estimation in fractional diffusion models,
volume 8 of *Bocconi & Springer Series*.
Springer, 2017.