Fractional Vasicek model: Asymptotic properties and statistical inference

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3. Statistical inference for fractional Ornstein–Uhlenbeck process
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Fractional Ornstein–Uhlenbeck process

Let $(\Omega, \mathfrak{F}, \mathrm{P})$ be a probability space, and $B^H = \{B_t^H, t \in \mathbb{R}\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ on this probability space. In this section we study the Langevin equation

$$X_t = x_0 + \theta \int_0^t X_s \, ds + B_t^H, \quad t \geq 0, \tag{1}$$

where $x_0 \in \mathbb{R}$, and $\theta \in \mathbb{R}$ is an unknown drift parameter. This equation has a unique solution:

$$X_t = x_0 e^{\theta t} + \theta e^{\theta t} \int_0^t e^{-\theta s} B_s^H \, ds + B_t^H, \quad t \geq 0. \tag{2}$$

The process $X = \{X_t, t \geq 0\}$ is called a fractional Ornstein–Uhlenbeck process.

Properties of variance (see Section 2)

- The random variable X_t has normal distribution $\mathcal{N}\left(x_0 e^{\theta t}, v(\theta, t)\right)$, with variance

$$v(\theta, t) = H \int_0^t s^{2H-1} \left(e^{\theta s} + e^{\theta(2t-s)}\right) ds. \qquad (3)$$

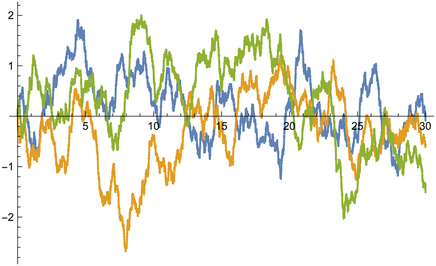
- Asymptotic behavior of variance

$$\text{If } \theta > 0, \text{ then } v(\theta, t) \sim \frac{H\Gamma(2H)}{\theta^{2H}} e^{2\theta t}, \text{ as } t \to \infty. \qquad (4)$$

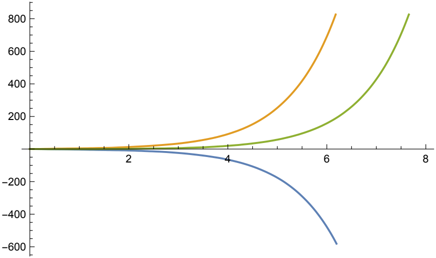
$$\text{If } \theta < 0, \text{ then } v(\theta, t) \to \frac{H\Gamma(2H)}{(-\theta)^{2H}}, \text{ as } t \to \infty. \qquad (5)$$

$$v(0, t) = t^{2H}, \quad t \geq 0. \qquad (6)$$

Ergodic case ($\theta < 0$) and Non-ergodic case ($\theta > 0$)



$\theta < 0$



$\theta > 0$

Least square estimator for the non-ergodic case $\theta > 0$

Assume that a trajectory of $X = X(t)$ is observed over a finite time interval $[0, T]$.

First we assume that $H \in (\frac{1}{2}, 1)$ and consider the least-square estimator of the following form:

$$\hat{\theta}_T^{(1)} = \frac{\int_0^T X_t \, dX_t}{\int_0^T X_t^2 \, dt}. \tag{7}$$

The estimator $\hat{\theta}_T^{(1)}$ admits the following representation:

$$\hat{\theta}_T^{(1)} = \frac{X_T^2 - x_0^2}{2 \int_0^T X_t^2 dt}. \tag{8}$$

Note that this form of the estimator is well defined for all $H \in (0, 1)$. In order to establish its strong consistency for all $H \in (0, 1)$, we need the following lemma.

Lemma 3.1

Let $H \in (0,1)$. *For* $\theta > 0$

$$e^{-\theta t} X_t \to \xi_\theta \quad a.\,s.,\ as\ t \to \infty,$$

where $\xi_\theta = x_0 + \theta \int_0^\infty e^{-\theta s} B_s^H \, ds \overset{d}{=} \mathcal{N}\left(x_0, \frac{H\Gamma(2H)}{\theta^{2H}}\right)$.

Proof. Note that the bound

$$\sup_{0 \le s \le t} \left| B_s^H \right| \le \left(\left(t^H (\log^+ t)^p \right) \vee 1 \right) \xi, \quad \text{a.s.} \tag{9}$$

implies the a. s. convergence $e^{-\theta t} B_t^H \to 0$, as $t \to \infty$. Therefore, by (2),

$$e^{-\theta t} X_t = x_0 + \theta \int_0^t e^{-\theta s} B_s^H \, ds + e^{-\theta t} B_t^H \to x_0 + \theta \int_0^\infty e^{-\theta s} B_s^H \, ds$$

a. s., as $t \to \infty$. It follows from (4) that the limit has the distribution $\mathcal{N}\left(x_0, \frac{H\Gamma(2H)}{\theta^{2H}}\right)$. □

Theorem 3.2

Let $H \in (0,1)$. For $\theta > 0$, the estimator $\hat{\theta}_T^{(1)}$ is strongly consistent, as $T \to \infty$.

Proof. Lemma 3.1 implies the a. s. convergence

$$\frac{X_T^2}{e^{2\theta T}} \to \xi_\theta^2, \quad \text{as } T \to \infty. \tag{10}$$

Therefore, by L'Hôpital's rule,

$$\lim_{T\to\infty} \frac{\int_0^T X_t^2\,dt}{e^{2\theta T}} = \lim_{T\to\infty} \frac{X_T^2}{2\theta e^{2\theta T}} = \frac{\xi_\theta^2}{2\theta}. \tag{11}$$

Note that $0 < \xi_\theta^2 < \infty$ with probability 1, since ξ_θ is a normal random variable. Combining (10) and (11), we get the convergence $\hat{\theta}_T^{(1)} \to \theta$ a. s., as $T \to \infty$. □

Remark 3.3

It was shown in [Belfadli et al.(2011), El Machkouri et al.(2016)] that under the assumptions of Theorem 3.2,

$$e^{\theta T}\left(\hat{\theta}_T^{(1)} - \theta\right) \xrightarrow{d} 2\theta\mathcal{C}(1),$$

as $T \to \infty$, where $\mathcal{C}(1)$ is the standard Cauchy distribution.

Rachid Belfadli, Khalifa Es-Sebaiy, and Youssef Ouknine.
Parameter estimation for fractional Ornstein–Uhlenbeck processes: non-ergodic case.
Frontiers in Science and Engineering, 1(1):1–16, 2011.

Mohamed El Machkouri, Khalifa Es-Sebaiy, and Youssef Ouknine.
Least squares estimator for non-ergodic Ornstein–Uhlenbeck processes driven by Gaussian processes.
J. Korean Statist. Soc., 45(3):329–341, 2016.

Ergodic-type estimator for the case $\theta < 0$

Let us consider an ergodic-type estimator proposed in [Hu and Nualart (2010)].
As before, assume that a trajectory of $X = X(t)$ is observed over a finite time interval $[0, T]$.

Theorem 3.4

Let $H \in (0,1)$. For $\theta < 0$, the estimator

$$\hat{\theta}_T^{(2)} = -\left(\frac{1}{\sigma^2 H \Gamma(2H) T} \int_0^T X_t^2 \, dt\right)^{-\frac{1}{2H}} \tag{12}$$

is strongly consistent, as $T \to \infty$.

Yaozhong Hu and David Nualart.
Parameter estimation for fractional Ornstein–Uhlenbeck processes.
Statistics and Probability Letters, 80(11-12):1030–1038, 2010.

Proof. Let Y be the stationary ergodic process, introduced in Section 2, i.e.,

$$Y_t = \theta\sigma e^{\theta t}\int_{-\infty}^{t} e^{-\theta s}B_s^H\,ds + \sigma B_t^H, \quad t \ge 0.$$

It follows from the ergodic theorem that

$$\frac{1}{T}\int_0^T Y_t^2\,dt \to \mathrm{E}Y_0^2,$$

as $T \to \infty$ a. s. and in L^2. The process X_t can be expressed as

$$X_t = Y_t - e^{\theta t}\eta_\theta, \tag{13}$$

where

$$\eta_\theta = \sigma\theta\int_{-\infty}^{0} e^{-\theta s}B_s^H\,ds - x_0$$

is a normal random variable. Using this representation, it is easy to show that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X_t^2\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T Y_t^2\,dt = \mathrm{E}Y_0^2 = \frac{\sigma^2 H\Gamma(2H)}{(-\theta)^{2H}},$$

where the last equality was established in the proof of Lemma 2.5. □

Remark 3.5

According to [Hu et al. (2019)], under the assumptions of Theorem 12, we have the following central limit theorem (for $H \le 3/4$) and noncentral limit theorem (for $H > 3/4$):

1. If $H \in (0, \frac{3}{4})$, then $\sqrt{T}\left(\hat\theta_T^{(2)} - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, -\frac{\theta\sigma_H^2}{(2H)^2}\right)$, as $T \to \infty$,

   $$\text{where} \quad \sigma_H^2 = \begin{cases} 4H - 1 + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(1-2H)\Gamma(2H)}, & H \in \left(0, \frac{1}{2}\right), \\ (4H-1)\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)\Gamma(2H)}\right), & H \in \left[\frac{1}{2}, \frac{3}{4}\right). \end{cases}$$

2. If $H = \frac{3}{4}$, then $\frac{\sqrt{T}}{\log T}\left(\hat\theta_T^{(2)} - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, -\frac{16\theta}{9\pi}\right)$, as $T \to \infty$.
3. If $H \in (\frac{3}{4}, 1)$, then $T^{2-2H}\left(\hat\theta_T^{(2)} - \theta\right) \xrightarrow{d} -\frac{(-\theta)^{2H-1}\sqrt{4H-2}}{\sqrt{4H-3}\,\Gamma(2H+1)}\zeta_{2H-1}$, as $T \to \infty$, where ζ_{2H-1} is a Rosenblatt random variable.

Yaozhong Hu, David Nualart, and Hongjuan Zhou.
Parameter estimation for fractional Ornstein–Uhlenbeck processes of general Hurst parameter.
Stat. Inference Stoch. Process., 22(1):111–142, 2019.

Hypothesis testing of the drift parameter sign

For hypothesis testing of the sign of the parameter θ we construct a test based on the asymptotic behavior of the random variable

$$Z(t) = \frac{\log^+ \log |X_t|}{\log t}, \qquad t > 1, \tag{14}$$

where $\log^+ x = \log x$ when $x > 1$ and $\log^+ x = 0$ otherwise.
The following result explains the main idea. It is based on the different asymptotic behavior of the fractional Ornstein–Uhlenbeck process with positive drift parameter and negative one.

Lemma 3.6

The value of $Z(t)$ converges a. s. to 1 for $\theta > 0$, and to 0 for $\theta \le 0$, as $t \to \infty$.

Proof. For $\theta > 0$, Lemma 3.1 implies the convergence

$$\log|X_t| - \theta t \to \log|\xi_\theta| \quad \text{a.s., as } t \to \infty,$$

where ξ_θ is a normal random variable, hence, $0 < |\xi_\theta| < \infty$ a.s. Therefore,

$$\frac{\log|X_t|}{t} \to \theta \quad \text{a.s., as } t \to \infty.$$

This means that there exists $\Omega' \subset \Omega$ such that $\mathrm{P}(\Omega') = 1$ and for any $\omega \in \Omega'$ there exists $t(\omega)$ such that for $t \geq t(\omega)$ $\log|X_t| > 0$. Hence, for $t \geq t(\omega)$ we have that

$$|Z(t) - 1| = \left|\frac{\log^+ \log|X_t|}{\log t} - 1\right| = \left|\frac{\log\log|X_t|}{\log t} - 1\right| = \left|\frac{\log\log|X_t| - \log t}{\log t}\right|$$
$$= \left|\frac{\log\frac{\log|X_t|}{t}}{\log t}\right| \to 0$$

a.s., as $t \to \infty$.

For $\theta \leq 0$, it is possible to derive from (9) the following bound[1]

$$\sup_{0 \leq u \leq s} |X_u| \leq \left(1 + s^H \log^2 s\right) \zeta,$$

where ζ nonnegative random variable with the following property: there exists $C > 0$ not depending on n such that $\mathrm{E} \exp\{x\zeta^2\} < \infty$ for any $0 < x < C$.
Hence,

$$|Z(t)| \leq \left| \frac{\log^+ \left(\log\left(1 + t^H \log^2 t\right) + \log \zeta\right)}{\log t} \right|$$
$$= \left| \frac{\log \left(\log\left(1 + t^H \log^2 t\right) + \log \zeta\right)}{\log t} \right| \sim \left| \frac{\log \left(\log\left(t^H \log^2 t\right)\right)}{\log t} \right| \to 0,$$

as $t \to \infty$. □

[1] K. Kubilius, Y. Mishura, K.R., O. Seleznjev. Consistency of the drift parameter estimator for the discretized fractional Ornstein–Uhlenbeck process with Hurst index $H \in (0, 1/2)$. *Electron. J. Statist.*, 9(2): 1799–1825, 2015.

CDF of the test statistic

The next result gives the cdf of $Z(t)$. Let Φ and φ denote the cdf and pdf, respectively, of the standard normal variable.

Lemma 3.7

For $t > 1$ the probability $g(\theta, x_0, t, c) = \mathrm{P}(Z(t) \le c)$ is given by

$$g(\theta, x_0, t, c) = \Phi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}}\right) + \Phi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta, t)}}\right) - 1, \qquad (15)$$

and g is a decreasing function of $\theta \in \mathbb{R}$.

Proof. Using (3) and taking into account that $\log^+ x$ is a non-decreasing function, we obtain

$$\begin{aligned} \mathrm{P}(Z(t) \le c) &= \mathrm{P}\left(|X_t| \le e^{t^c}\right) \\ &= \Phi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}}\right) - \Phi\left(\frac{-e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}}\right) \\ &= \Phi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}}\right) + \Phi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta, t)}}\right) - 1. \end{aligned}$$

Let us prove the monotonicity of the function g with respect to θ. Note that g is an even function with respect to x_0. Therefore, it suffices to consider only the case $x_0 \ge 0$.

The partial derivative equals

$$\frac{\partial g}{\partial \theta} = \varphi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right)$$
$$\times\left(-x_0 t e^{\theta t} v^{-\frac{1}{2}}(\theta,t) - \frac{1}{2} v^{-\frac{3}{2}}(\theta,t) v'_\theta(\theta,t)\left(e^{t^c} - x_0 e^{\theta t}\right)\right)$$
$$+\varphi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right)$$
$$\times\left(x_0 t e^{\theta t} v^{-\frac{1}{2}}(\theta,t) - \frac{1}{2} v^{-\frac{3}{2}}(\theta,t) v'_\theta(\theta,t)\left(e^{t^c} + x_0 e^{\theta t}\right)\right)$$
$$= -\frac{1}{2} v^{-\frac{3}{2}}(\theta,t) v'_\theta(\theta,t) e^{t^c}\left(\varphi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) + \varphi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right)\right)$$
$$- x_0 e^{\theta t} v^{-\frac{3}{2}}(\theta,t)\left(t v(\theta,t) - \frac{1}{2} v'_\theta(\theta,t)\right)$$
$$\times\left(\varphi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right) - \varphi\left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta,t)}}\right)\right). \tag{16}$$

Since

$$v'_\theta(\theta, t) = H \int_0^t s^{2H-1} \left(s e^{\theta s} + (2t - s) e^{\theta(2t-s)} \right) ds > 0, \qquad (17)$$

we see that the first term in the left-hand side of (16) is negative. Let us consider the second term. From (3) and (17) it follows that

$$t v(\theta, t) - \tfrac{1}{2} v'_\theta(\theta, t) = H \int_0^t s^{2H-1} \left(\left(t - \tfrac{1}{2} s\right) e^{\theta s} + \tfrac{1}{2} s e^{\theta(2t-s)} \right) ds > 0.$$

Since $\left| e^{t^c} - x_0 e^{\theta t} \right| \le e^{t^c} + x_0 e^{\theta t}$ for $x_0 \ge 0$, we have

$$\varphi\left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) - \varphi\left(\frac{e^{t^c} + x_0 e^{\theta_1 t}}{\sqrt{v(\theta, t)}} \right) \ge 0.$$

Thus, the second term in the left-hand side of (16) is non-positive. Hence, $\frac{\partial g}{\partial \theta} < 0$. □

Testing $H_0\colon \theta \le 0$ against $H_1\colon \theta > 0$

We consider the test with the following procedure of testing the hypothesis $H_0\colon \theta \le 0$ against the alternative $H_1\colon \theta > 0$. For a given significance level α, and for sufficiently large value of t we choose a threshold $c = c_t \in (0,1)$. Further, when $Z(t) \le c$ the hypothesis H_0 is accepted, and when $Z(t) > c$ it is rejected. Below we will propose a technically simpler version of this test, without the computation of c. The threshold c can be chosen as follows.

Fix a number $\alpha \in (0,1)$, the significance level of the test. This level gives the maximal probability of a type I error, that is in our case the probability to reject the hypothesis $H_0\colon \theta \le 0$ when it is true. By Lemma 3.7, for a threshold $c \in (0,1)$ and $t > 1$ this probability equals

$$\sup_{\theta \le 0} \mathrm{P}(Z(t) > c) = 1 - g(0, x_0, t, c).$$

Therefore, we determine c_t as a solution to the equation

$$g(0, x_0, t, c_t) = 1 - \alpha. \tag{18}$$

The following result shows that for any $\alpha \in (0,1)$, it is possible to choose a sufficiently large t such that $c_t \in (0,1)$.

Lemma 3.8

Let $\alpha \in (0,1)$. Then there exists $t_0 \geq 1$ such that for all $t > t_0$ there exists a unique $c_t \in (0,1)$ such that $g(0, x_0, t, c_t) = 1 - \alpha$. Moreover $c_t \to 0$, as $t \to \infty$.
The constant t_0 can be chosen as the largest $t \geq 1$ that satisfies at least one of the following two equalities

$$g(0, x_0, t, 0) = 1 - \alpha \quad \text{or} \quad g(0, x_0, t, 1) = 1 - \alpha. \tag{19}$$

Proof. By (6), $v(0,t) = t^{2H}$. Then for $\theta = 0$ the formula (15) becomes

$$g(0, x_0, t, c) = \Phi\left(\frac{e^{t^c} - x_0}{t^H}\right) + \Phi\left(\frac{e^{t^c} + x_0}{t^H}\right) - 1. \tag{20}$$

For any $t > 1$, the function $g(0, x_0, t, c)$ is strictly increasing with respect to c.

For $c = 0$ we have

$$g(0, x_0, t, 0) = \Phi\left(\frac{e - x_0}{t^H}\right) + \Phi\left(\frac{e + x_0}{t^H}\right) - 1 \to 2\Phi(0) - 1 = 0, \text{ as } t \to \infty.$$

Therefore, there exists $t_1 > 1$ such that $g(0, x_0, t, 0) < 1 - \alpha$ for all $t \geq t_1$. Similarly, for $c = 1$

$$g(0, x_0, t, 1) = \Phi\left(\frac{e^t - x_0}{t^H}\right) + \Phi\left(\frac{e^t + x_0}{t^H}\right) - 1 \to 2\Phi(\infty) - 1 = 1, \ t \to \infty.$$

Therefore, there exists $t_2 > 1$ such that $g(0, x_0, t, 1) > 1 - \alpha$ for all $t \geq t_2$. Thus, for any $t \geq t_0 = \max\{t_1, t_2\}$ there exists a unique $c_t \in (0, 1)$ such that $g(0, x_0, t, c_t) = 1 - \alpha$.

To prove the convergence $c_t \to 0$, $t \to \infty$, consider an arbitrary $\varepsilon \in (0,1)$. Then

$$g(0, x_0, t, \varepsilon) = \Phi\left(\frac{e^{t^\varepsilon} - x_0}{t^H}\right) + \Phi\left(\frac{e^{t^\varepsilon} + x_0}{t^H}\right) - 1 \to 2\Phi(\infty) - 1 = 1,$$

as $t \to \infty$. Arguing as above, we see that there exists $t_3 > 1$ such that for any $t > t_3$ the unique $c_t \in (0,1)$, for which $g(0, x_0, t, c_t) = 1 - \alpha$, belongs to the interval $(0, \varepsilon)$. This implies the convergence $c_t \to 0$, as $t \to \infty$.
It follows from (20) that $g(0, x_0, t, 0) = g(0, x_0, t, 1)$ for $t = 1$. As $t \to \infty$, we have $g(0, x_0, t, 0) \to 0$, $g(0, x_0, t, 1) \to 1$. Hence, at least one of the equalities (19) is satisfied for some $t \geq 1$ and the set of such t's is bounded. □

Remark 3.9

Since the function $g(0, x_0, t, c)$ is strictly increasing with respect to c for $t > 1$, we see that the inequality $Z(t) \le c_t$ is equivalent to the inequality $g(0, x_0, t, Z(t)) \le g(0, x_0, t, c_t) = 1 - \alpha$. Therefore, we do not need to compute the value of c_t. It is sufficient to compare $g(0, x_0, t, Z(t))$ with the level $1 - \alpha$.

Algorithm 1

The hypothesis $H_0 \colon \theta \le 0$ against the alternative $H_1 \colon \theta > 0$ can be tested as follows.

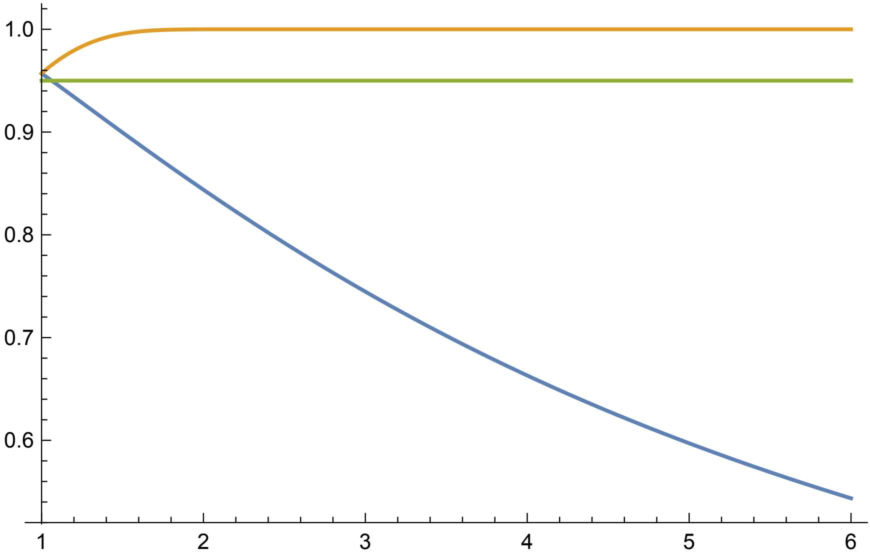
1. Find t_0 defined in Lemma 3.8. The algorithm can be applied only in the case $t > t_0$.
2. Evaluate the statistic $Z(t)$ defined by (14).
3. Compute the value of $g(0, x_0, t, Z(t))$, see (20).
4. Do not reject the hypothesis H_0 if $g(0, x_0, t, Z(t)) \le 1 - \alpha$, and reject it otherwise.

Remark 3.10

In fact, the condition $t > t_0$ is not too restrictive, since for reasonable values of α, the values of t_0 are quite small, see Table 1.

Table: Value of t_0 for various H and α

H	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\alpha = 0.01$	1.216	1.231	1.249	1.270	1.294	1.322	1.356	1.397	1.446
$\alpha = 0.05$	1.531	1.237	1.153	1.112	1.089	1.074	1.063	1.055	1.049


1.0
0.9
0.8
0.7
0.6
1
2
3
4
5
6

Let us summarize the properties of the test in the following theorem.

Theorem 3.11

The test described in Algorithm 1 is unbiased and consistent, as $t \to \infty$*. For a simple alternative* $\theta_1 > 0$ *and moment* $t > t_0$*, the power of the test equals* $1 - g(\theta_1, x_0, t, c_t)$*, where* c_t *can be found from* (18)*.*

Proof. It follows from the monotonicity of g with respect to θ (see Lemma 3.7) that for any $\theta_1 > 0$

$$\mathrm{P}(Z(t) > c_t) = 1 - g(\theta_1, x_0, t, c_t) > 1 - g(0, x_0, t, c_t) = \alpha.$$

This means that the test is unbiased. Evidently, for a simple alternative $\theta_1 > 0$ the power of the test equals $1 - g(\theta_1, x_0, t, c_t)$.

It follows from the convergence $c_t \to 0$, as $t \to \infty$ (see Lemma 3.8), that $c_t < c$ for sufficiently large t and some constant $c \in (0,1)$. Taking into account the formula (15) and the convergence (4), we get, as $t \to \infty$:

$$\begin{aligned} 1 &\geq 1 - g(\theta_1, x_0, t, c_t) \geq 1 - g(\theta_1, x_0, t, c) \\ &= 2 - \Phi\left(\frac{e^{t^c} - x_0 e^{\theta_1 t}}{\sqrt{v(\theta_1, t)}}\right) - \Phi\left(\frac{e^{t^c} + x_0 e^{\theta_1 t}}{\sqrt{v(\theta_1, t)}}\right) \\ &\to 2 - \Phi\left(-\frac{x_0 \theta_1^H}{\sqrt{H\Gamma(2H)}}\right) - \Phi\left(\frac{x_0 \theta_1^H}{\sqrt{H\Gamma(2H)}}\right) = 1. \end{aligned}$$

Hence, the test is consistent. □

Numerical illustrations

Table: Empirical rejection probabilities of the test for the hypothesis testing $H_0 \colon \theta \le 0$ against the alternative $H_1 \colon \theta > 0$ for $H = 0.3$ and $H = 0.7$

θ	-0.1	-0.05	0	0.05	0.1	0.15	0.2	0.25	0.3
				$\boldsymbol{H = 0.3}$					
$T = 20$	0.000	0.003	0.043	0.341	0.701	0.880	0.973	0.982	0.996
$T = 40$	0.000	0.000	0.043	0.675	0.952	0.995	0.999	1.000	1.000
$T = 60$	0.000	0.000	0.039	0.860	0.994	1.000	1.000	1.000	1.000
$T = 80$	0.000	0.000	0.048	0.940	1.000	1.000	1.000	1.000	1.000
$T = 100$	0.000	0.000	0.049	0.986	1.000	1.000	1.000	1.000	1.000
				$\boldsymbol{H = 0.7}$					
$T = 20$	0.000	0.001	0.058	0.284	0.540	0.800	0.910	0.967	0.979
$T = 40$	0.000	0.000	0.050	0.581	0.889	0.984	0.998	1.000	1.000
$T = 60$	0.000	0.000	0.042	0.782	0.980	1.000	0.999	1.000	1.000
$T = 80$	0.000	0.000	0.047	0.908	0.995	1.000	1.000	1.000	1.000
$T = 100$	0.000	0.000	0.048	0.959	1.000	1.000	1.000	1.000	1.000

3 Statistical inference for fractional Ornstein–Uhlenbeck process

4 Maximum likelihood estimation in fractional Vasicek model
- Auxiliary results: exact and asymptotic distributions of statistics
- Main results: asymptotic distributions of the estimators

The problem

We study the fractional Vasicek model, described by the stochastic differential equation

$$X_t = x_0 + \int_0^t (\alpha - \beta X_s)\, ds + \gamma B_t^H, \quad t \geq 0. \tag{21}$$

The main goal is to estimate parameters α and β by continuous observations of a trajectory of X on the interval $[0, T]$.

The problem

We study the fractional Vasicek model, described by the stochastic differential equation

$$X_t = x_0 + \int_0^t (\alpha - \beta X_s)\, ds + \gamma B_t^H, \quad t \geq 0. \tag{21}$$

The main goal is to estimate parameters α and β by continuous observations of a trajectory of X on the interval $[0, T]$.

We assume that the parameters $\gamma > 0$ and $H \in (1/2, 1)$ are known. This assumption is natural, because γ and H can be obtained explicitly from the observations by considering realized power variations, see Remark 4.1 below.

Equation (21) has a unique solution, which is given by

$$X_t = x_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \gamma \int_0^t e^{-\beta(t-s)} dB_s^H, \quad t \geq 0. \tag{22}$$

Estimation of β when $\alpha = 0$

Fractional Ornstein–Uhlenbeck process:

$$X_t = x_0 - \beta \int_0^t X_s ds + \gamma B_t^H, \quad t \geq 0.$$

- 2002 Kleptsyna, Le Breton (MLE, strong consistency, $H > 1/2$)
- 2007 Tudor, Viens (MLE, strong consistency, $H < 1/2$)
- 2008 Bishwal (MCE)
- 2010 Bercu, Coutin, Savy (MLE, asymptotic normality, $\beta > 0$)
- 2010 Hu, Nualart (ergodic-type estimator, $\beta > 0$)
- 2011 Belfadli, Es-Sebaiy, Ouknine (LSE, $\beta < 0$)
- 2012 Moers (combination of ergodic-type and LSE)
- 2013 Hu, Song (discrete ergodic-type estimator, $\beta > 0$)
- 2013 Tanaka (MLE and MCE, exact distributions, $\beta > 0$)
- 2015 Tanaka (MLE, exact and asymptotic distribution, $\beta < 0$)
- 2017 Hu, Nualart, Zhou (ergodic-type estimator, $\beta > 0$)
- 2021 Haress, Hu (ergodic-type estimator for 3 parameters, $\beta > 0$)

Notation

For $0 < s < t \le T$ we define

$$\kappa_H = 2H\Gamma\left(3/2 - H\right)\Gamma\left(H + 1/2\right), \qquad \lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)},$$

$$k_H(t,s) = \kappa_H^{-1} s^{1/2-H}(t-s)^{1/2-H}, \qquad w_t^H = \lambda_H^{-1} t^{2-2H}.$$

We introduce also three stochastic processes

$$P_H(t) = \frac{1}{\gamma}\frac{d}{dw_t^H}\int_0^t k_H(t,s) X_s\,ds,$$

$$Q_H(t) = \frac{1}{\gamma}\frac{d}{dw_t^H}\int_0^t k_H(t,s)(\alpha - \beta X_s)\,ds = \frac{\alpha}{\gamma} - \beta P_H(t),$$

$$S_t = \frac{1}{\gamma}\int_0^t k_H(t,s)\,dX_s.$$

Semimartingale property

According to [Kleptsyna et al. (2000), Theorem 1], the process S is an $(\mathfrak{F}_t)$-semimartingale with the decomposition

$$S_t = \int_0^t Q_H(s)\,dw_s^H + M_t^H, \tag{23}$$

where $M_t^H = \int_0^t k_H(t,s)\,dB_s^H$ is a Gaussian martingale with variance function $\langle M^H \rangle = w^H$. Natural filtrations of processes S and X coincide. Moreover, the process X admits the following representation

$$X_t = \int_0^t K_H(t,s)\,dS_s, \tag{24}$$

where $K_H(t,s) = \gamma H(2H-1)\int_s^t r^{H-1/2}(r-s)^{H-3/2}\,dr$.

M. L. Kleptsyna, A. Le Breton, and M.-C. Roubaud.
Parameter estimation and optimal filtering for fractional type stochastic systems.
Stat. Inference Stoch. Process., 3(1-2):173–182, 2000.

Remark 4.1

If we observe the whole path $\{X_t, t \in [0, T]\}$, then the parameters γ and H can be obtained from observations explicitly in the following way. Let $\left\{t_i^{(n)}\right\}$ be a partition of $[0, T]$, such that $\sup_i \left|t_{i+1}^{(n)} - t_i^{(n)}\right| \to 0$, as $n \to \infty$. Denote $Z_T = \int_0^T k_H(T, s)\, dX_s = \gamma S_T$. From (23) it follows that $\langle Z \rangle_T = \gamma^2 w_T^H$ a. s. Hence, the parameter γ is calculated as the limit

$$\gamma^2 = \left(w_T^H\right)^{-1} \lim_n \sum_i \left(Z_{t_{i+1}^{(n)}} - Z_{t_i^{(n)}}\right)^2 \quad \text{a. s..}$$

The Hurst index H can be evaluated as follows:

$$H = \frac{1}{2} - \frac{1}{2} \lim_{n} \log_2 \left(\frac{\sum_{i=1}^{2n-1} \left(X_{t_{i+1}^{(2n)}} - 2X_{t_i^{(2n)}} + X_{t_{i-1}^{(2n)}} \right)^2}{\sum_{i=1}^{n-1} \left(X_{t_{i+1}^{(n)}} - 2X_{t_i^{(n)}} + X_{t_{i-1}^{(n)}} \right)^2} \right) \quad \text{a. s.},$$

see, e. g., [Kubilius et al.(2017), Sec. 3.1]. There exist several other methods of the Hurst index evaluation based on power variations of X.

Kęstutis Kubilius, Yuliya Mishura, and Kostiantyn Ralchenko.
Parameter estimation in fractional diffusion models, volume 8 of *Bocconi & Springer Series*.
Springer, 2017.

Likelihood function

Applying the analog of the Girsanov formula for a fractional Brownian motion [Kleptsyna et al. (2000), Theorem 3] and (23), one can obtain the likelihood ratio $\frac{d\mathrm{P}_{\alpha,\beta}(T)}{d\mathrm{P}_{0,0}(T)}$ for the probability measure $\mathrm{P}_{\alpha,\beta}(T)$ corresponding to our model and probability measure $\mathrm{P}_{0,0}(T)$ corresponding to the model with zero drift:

$$\begin{aligned}\frac{d\mathrm{P}_{\alpha,\beta}(T)}{d\mathrm{P}_{0,0}(T)} = \exp\Big\{ & \frac{\alpha}{\gamma} S_T - \beta \int_0^T P_H(t)\, dS_t - \frac{\alpha^2}{2\gamma^2} w_T^H \\ & + \frac{\alpha\beta}{\gamma} \int_0^T P_H(t)\, dw_t^H - \frac{\beta^2}{2} \int_0^T (P_H(t))^2 \, dw_t^H \Big\}. \end{aligned} \tag{25}$$

Maximum likelihood estimators

MLEs of parameters α and β maximize (25) and have the following form:

$$\widehat{\alpha}_T = \frac{S_T K_T - I_T J_T}{w_T^H K_T - J_T^2}\,\gamma, \qquad \widehat{\beta}_T = \frac{S_T J_T - w_T^H I_T}{w_T^H K_T - J_T^2}, \tag{26}$$

where

$$\begin{aligned}
S_T &= \frac{1}{\gamma}\int_0^T k_H(T,s)\,dX_s,\\
I_T &= \int_0^T P_H(t)\,dS_t,\\
J_T &= \int_0^T P_H(t)\,dw_t^H = \frac{1}{\gamma}\int_0^T k_H(T,s)X_s\,ds,\\
K_T &= \int_0^T (P_H(t))^2\,dw_t^H.
\end{aligned}$$

Exact joint distribution of the statistics S_T, I_T, J_T, K_T

Lemma 4.2 (Joint distribution of S_T and I_T)

Moment generating function of (S_T, I_T) *equals*

$$m_1^{(\alpha,\beta)}(\xi_1,\xi_2) = \mathrm{E}\left[\exp\left\{\xi_1 S_T + \xi_2 I_T\right\}\right] = D^{(\alpha,\beta)}(\xi_2)^{-\frac{1}{2}} \exp\left\{\frac{1}{8D^{(\alpha,\beta)}(\xi_2)}\sum_{i=1}^{4} A_i^{(\alpha,\beta)}(\xi_1,\xi_2) - \frac{\xi_2 T}{2}\right\},$$

where[a]

$$D^{(\alpha,\beta)}(\xi_2) = \left(1-\frac{\xi_2}{2\beta}\right)^2 + \frac{\xi_2^2}{4\beta^2}e^{-2\beta T} + \left(\frac{\xi_2}{\beta} - \frac{\xi_2^2}{2\beta^2}\right)\frac{\beta\pi T}{4\sin\pi H}e^{-\beta T} \times \left[I_{-H}\left(\frac{\beta T}{2}\right)I_{H-1}\left(\frac{\beta T}{2}\right) + I_{1-H}\left(\frac{\beta T}{2}\right)I_H\left(\frac{\beta T}{2}\right)\right], \quad (27)$$

[a] $I_\nu(x)$ is the modified Bessel function of the first kind (see below)

Lemma 4.2 (cont.)

$$A_1^{(\alpha,\beta)}(\xi_1,\xi_2) = \xi_2\left(c_1\left(\tfrac{\alpha}{\beta}\right)\xi_1 - c_2\left(\tfrac{\alpha}{\beta}\right)\xi_2\right)\beta^{H-1}T^{1-H}e^{-\frac{3\beta T}{2}}I_{1-H}\left(\frac{\beta T}{2}\right),$$

$$A_2^{(\alpha,\beta)}(\xi_1,\xi_2) = \left(\xi_1^2 c_3 - \xi_1\xi_2 c_4\left(\tfrac{\alpha}{\beta}\right) + \xi_2^2 c_5\left(\tfrac{\alpha}{\beta}\right)\right) \times T^{2-2H}e^{-\beta T}I_{1-H}\left(\frac{\beta T}{2}\right)I_{H-1}\left(\frac{\beta T}{2}\right),$$

$$A_3^{(\alpha,\beta)}(\xi_1,\xi_2) = \xi_2(\xi_2 - 2\beta)\,c_6\left(\tfrac{\alpha}{\beta}\right)\beta^{2H-1}Te^{-\beta T}I_{1-H}\left(\frac{\beta T}{2}\right)I_{-H}\left(\frac{\beta T}{2}\right),$$

$$A_4^{(\alpha,\beta)}(\xi_1,\xi_2) = \left(c_1\left(\tfrac{\alpha}{\beta}\right)\xi_1 - c_2\left(\tfrac{\alpha}{\beta}\right)\xi_2\right)(\xi_2 - 2\beta)\,\beta^{H-1} \times T^{1-H}e^{-\frac{\beta T}{2}}I_{1-H}\left(\frac{\beta T}{2}\right),$$

Lemma 4.2 (cont.)

$$c_1\left(\tfrac{\alpha}{\beta}\right) = \left(x_0 - \tfrac{\alpha}{\beta}\right)4\rho_H, \qquad c_4\left(\tfrac{\alpha}{\beta}\right) = \left(x_0 - \tfrac{\alpha}{\beta}\right)\rho_H 2^{2H+1}\Gamma(H),$$

$$c_2\left(\tfrac{\alpha}{\beta}\right) = \left(x_0 - \tfrac{\alpha}{\beta}\right)^2 \frac{\lambda_H^* 2^{2H+1}\rho_H^2}{\Gamma(1-H)}, \quad c_5\left(\tfrac{\alpha}{\beta}\right) = \left(x_0 - \tfrac{\alpha}{\beta}\right)^2 \frac{\lambda_H^* 2^{4H-1}\rho_H^2\Gamma(H)}{\Gamma(1-H)},$$

$$c_3 = \frac{2\Gamma(H)\Gamma(1-H)}{\lambda_H^*}, \qquad c_6\left(\tfrac{\alpha}{\beta}\right) = \left(x_0 - \frac{\alpha}{\beta}\right)^2 2\lambda_H^*\rho_H^2,$$

$$\lambda_H^* = \frac{\lambda_H}{2-2H}, \qquad \rho_H = \frac{\sqrt{\pi}\,\Gamma(3/2-H)}{\gamma\kappa_H}.$$

Domain of the function $m_1^{(\alpha,\beta)}$ equals $\left\{(\xi_1,\xi_2)\in\mathbb{R}^2 : D^{(\alpha,\beta)}(\xi_2) > 0\right\}$.

Modified Bessel function of the first kind

Let $\nu > -1$, $x \in \mathbb{R}$. Then the function $I_\nu(x)$ could be defined by the following power series:

$$I_\nu(x) = \sum_{j=0}^{\infty} \frac{(x/2)^{2j+\nu}}{j!\Gamma(j+1+\nu)}.$$

Note, that if x is negative and ν is non-integer, then the function $I_\nu(x)$ is complex-valued. However, a function $I_\nu(x)/x^\nu$ is always real-valued. This function equals $2^{-\nu}/\Gamma(1+\nu)$ when $x = 0$ and it is even, i.e.

$$\frac{I_\nu(-x)}{(-x)^\nu} = \frac{I_\nu(x)}{x^\nu}. \tag{28}$$

For $\nu > -\frac{1}{2}$ the function $I_\nu(x)$ can alternatively be defined with the help of the integral:

$$I_\nu(x) = \frac{(x/2)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_{-1}^{1} \left(1 - t^2\right)^{\nu - \frac{1}{2}} e^{-xt}\, dt. \tag{29}$$

For large values of x the function $I_\nu(x)$ has the following asymptotic behavior:

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \left(1 - \frac{4\nu^2 - 1}{8x} + O\left(x^{-2}\right)\right), \qquad x \to \infty. \tag{30}$$

The following lemma gives a joint MGF of (S_T, I_T, J_T, K_T).

Lemma 4.3

Moment generating function of (S_T, I_T, J_T, K_T) *equals*

$$m_2(\theta_1, \theta_2, \theta_3, \theta_4) = \mathrm{E}\left[\exp\left\{\theta_1 S_T + \theta_2 I_T + \theta_3 J_T + \theta_4 K_T\right\}\right]$$
$$= m_1^{(\alpha_1,\beta_1)}\left(\theta_1 + \frac{\alpha - \alpha_1}{\gamma}, \theta_2 - \beta + \beta_1\right) \exp\left\{\frac{\alpha_1^2 - \alpha^2}{2\gamma^2} w_T^H\right\},$$

where

$$\alpha_1 = \frac{\gamma\theta_3 + \alpha\beta}{\sqrt{\beta^2 - 2\theta_4}}, \qquad \beta_1 = \sqrt{\beta^2 - 2\theta_4}.$$

Domain of the function m_2 *equals*

$$\left\{(\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{R}^4 : \theta_4 < \frac{\beta^2}{2}, D^{(\alpha,\beta)}\left(\theta_2 - \beta + \sqrt{\beta^2 - 2\theta_4}\right) > 0\right\},$$

where $D^{(\alpha,\beta)}$ *is defined in* (27).

Asymptotic behavior of statistics in the case $\beta > 0$

Lemma 4.4 (Asymptotic behavior of statistics in the case $\beta > 0$)

Let $\beta > 0$. *As* $T \to \infty$,

$$
\begin{aligned}
T^{H-\frac{1}{2}} S_T &\xrightarrow{d} \mathcal{N}\left(-\frac{c_1 \beta^{H-1/2}}{4\sqrt{\pi}}, \frac{c_3}{4\pi\beta}\right), \\
\frac{I_T}{T} &\xrightarrow{\mathrm{P}} -\frac{1}{2}, \\
\frac{J_T}{w_T^H} &\xrightarrow{\mathrm{P}} \frac{\alpha}{\beta\gamma}, \\
\frac{K_T}{T} &\xrightarrow{\mathrm{P}} \frac{1}{2\beta}.
\end{aligned}
$$

Define

$$V_1(T) = \frac{1}{\sqrt{w_T^H}} \left(J_T - \frac{\alpha}{\beta\gamma} w_T^H \right),$$

$$V_2(T) = \frac{1}{\sqrt{T}} \left(I_T - \frac{\alpha}{\gamma} J_T + \beta K_T \right).$$

Lemma 4.5

Let $\beta > 0$*. As* $T \to \infty$*,*

$$\mathrm{E} \exp \{\mu_1 V_1(T) + \mu_2 V_2(T)\} \to \exp \left\{ \frac{\mu_1^2}{2\beta^2} + \frac{\mu_2^2}{4\beta} \right\}, \tag{31}$$

i. e.,

$$\begin{bmatrix} V_1(T) \\ V_2(T) \end{bmatrix} \xrightarrow{d} \mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\beta^2} & 0 \\ 0 & \frac{1}{2\beta} \end{bmatrix} \right). \tag{32}$$

Asymptotic behavior of statistics in the case $\beta < 0$

Lemma 4.6

Under stated conditions the process S_T has normal asymptotic distribution as $T \to \infty$, namely

$$T^{H-1/2} e^{\beta T} S_T \xrightarrow{d} \mathcal{N}\left(\frac{\left(x_0 - \frac{\alpha}{\beta}\right) \rho_H(-\beta)^{H-1/2}}{\sqrt{\pi}}, \frac{\Gamma(H)\Gamma(1-H)}{2\pi(-\beta)\lambda_H^*} \right). \quad (33)$$

The following result is crucial for the derivation of the joint asymptotic distribution of MLE.

Lemma 4.7

Vector of main components of the MLE have the following behaviour

$$\begin{pmatrix} T^{H-1}\left(S_T + \beta J_T - \frac{\alpha}{\gamma} w_T^H\right) \\ e^{\beta T}(I_T + \beta K_T) \\ e^{2\beta T} K_T \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \xi \\ \eta\zeta \\ \zeta^2 \end{pmatrix}, \quad \text{as } T \to \infty, \tag{34}$$

where ξ, η, ζ *are independent and* $\xi \overset{d}{=} \mathcal{N}\left(0, \lambda_H^{-1}\right)$, $\eta \overset{d}{=} \mathcal{N}(0,1)$,

$$\zeta \overset{d}{=} \mathcal{N}\left(\frac{\left(x_0 - \frac{\alpha}{\beta}\right) \rho_H \sqrt{\lambda_H^*}(-\beta)^{H-1}}{\sqrt{2\pi}}, \frac{1}{4\beta^2 \sin \pi H}\right). \tag{35}$$

Remark 4.8

In fact, $\mathcal{N}\left(0, \lambda_H^{-1}\right)$ is an exact distribution of the random variable $T^{H-1}\big(S_T + \beta J_T - \frac{\alpha}{\gamma} w_T^H\big)$ for any T.

The following series of corollaries will describe asymptotic distributions of minor components of the MLE.
First, by considering the convergence of the first component of the random vector in (34), we immediately get the following result.

Corollary 4.9

For process $(S_T + \beta J_T)$ *we have*

$$\frac{1}{w_T^H}(S_T + \beta J_T) \xrightarrow{\mathrm{P}} \frac{\alpha}{\gamma}, \qquad T \to \infty.$$

Corollary 4.10

For process I_T we have

$$e^{2\beta T} I_T \xrightarrow{d} -\beta\zeta^2, \qquad T \to \infty,$$

where ζ has the normal distribution defined in (35).

Corollary 4.11

For process J_T we have

$$T^{H-\frac{1}{2}} e^{\beta T} J_T \xrightarrow{d} \mathcal{N}\left(\frac{8(x_0 - \frac{\alpha}{\beta})\rho_H(-\beta)^{H-3/2}}{\sqrt{\pi}}, \frac{4\Gamma(H)\Gamma(1-H)}{\lambda_H^*(-\beta)^3\pi}\right), \ T \to \infty.$$

Ergodic case $\beta > 0$

Theorem 4.12 (Main result for $\beta > 0$)

Let $\beta > 0$, $H \in (1/2, 1)$. Then the MLE $\left(\widehat{\alpha}_T, \widehat{\beta}_T\right)$ of the parameter (α, β) is asymptotically normal:

$$\begin{bmatrix} T^{1-H}\left(\widehat{\alpha}_T - \alpha\right) \\ \sqrt{T}\left(\widehat{\beta}_T - \beta\right) \end{bmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \lambda_H \gamma^2 & 0 \\ 0 & 2\beta \end{bmatrix}\right), \quad T \to \infty, \tag{36}$$

that is, the estimators $\widehat{\alpha}_T$ and $\widehat{\beta}_T$ are asymptotically independent.

Remark 4.13 (Alternative parameterization)

An alternative specification of the fractional Vasicek model is

$$dX_t = \kappa\,(\mu - X_t)\,dt + \gamma dB_t^H, \quad t \in [0, T], \quad X_0 = x_0.$$

The MLEs of the parameters μ and κ have the form

$$\widehat{\mu}_T = \frac{S_T K_T - I_T J_T}{S_T J_T - w_T^H I_T}\,\gamma, \qquad \widehat{\kappa}_T = \frac{S_T J_T - w_T^H I_T}{w_T^H K_T - J_T^2}.$$

If $\kappa > 0$ and $H \in (1/2, 1)$, then

$$\begin{bmatrix} T^{1-H}\,(\widehat{\mu}_T - \mu) \\ \sqrt{T}\,(\widehat{\kappa}_T - \kappa) \end{bmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\lambda_H \gamma^2}{\kappa^2} & 0 \\ 0 & 2\kappa \end{bmatrix} \right),$$

The proof is carried out by delta-method.

Now let us consider the situation when one of the parameters is known. In this case we can obtain strong consistency of the corresponding MLEs (instead of weak one) by applying the strong law of large numbers for martingales.

Theorem 4.14

Let $\beta > 0$ be known and $H \in (1/2, 1)$. The MLE for α is

$$\widetilde{\alpha}_T = \frac{\gamma}{w_T^H}\left(S_T + \beta J_T\right).$$

It is unbiased, strongly consistent and normal:

$$T^{1-H}\left(\widetilde{\alpha}_T - \alpha\right) \stackrel{d}{=} \mathcal{N}\left(0, \lambda_H \gamma^2\right).$$

Theorem 4.15

Let α *be known,* $H \in (1/2, 1)$ *and* $\beta > 0$*. The MLE for* β *is*

$$\widetilde{\beta}_T = \frac{\frac{\alpha}{\gamma} J_T - I_T}{K_T}.$$

It is strongly consistent and asymptotically normal:

$$\sqrt{T}\left(\widehat{\beta}_T - \beta\right) \xrightarrow{d} \mathcal{N}(0, 2\beta)$$

Non-ergodic case $\beta < 0$

Theorem 4.16 (Main result for $\beta < 0$)

Let $\beta < 0$, $H \in (1/2, 1)$. *Then*

$$\begin{pmatrix} T^{1-H}\left(\widehat{\alpha}_T - \alpha\right) \\ e^{-\beta T}\left(\widehat{\beta}_T - \beta\right) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \nu \\ \frac{\eta}{\zeta} \end{pmatrix}, \qquad T \to \infty, \tag{37}$$

where $\nu \overset{d}{=} \mathcal{N}(0, \lambda_H \gamma^2)$, $\eta \overset{d}{=} \mathcal{N}(0,1)$, *and*

$$\zeta \overset{d}{=} \mathcal{N}\left(\frac{(x_0 - \frac{\alpha}{\beta})\rho_H \sqrt{\lambda_H^*}(-\beta)^{H-1}}{\sqrt{2\pi}}, \frac{1}{4\beta^2 \sin \pi H}\right) \tag{38}$$

are independent random variables. In particular, the estimators $\widehat{\alpha}_T$ *and* $\widehat{\beta}_T$ *are asymptotically independent.*

Remark 4.17

Unlike the ergodic case, in the non-ergodic case the initial value x_0 affects the asymptotic bias of $\widehat{\beta}_T$. If $\beta < 0$, then the deterministic term $(x_0 - \frac{\alpha}{\beta})e^{-\beta t}$ in (22) does not converge to zero and, moreover, has the same asymptotic order $O(e^{-\beta T})$ as the stochastic term $\gamma \int_0^t e^{-\beta(t-s)} dB_s^H$. This implies that the asymptotic behavior of statistics S_T, I_T, J_T, and K_T depends on x_0. Similar dependence on initial condition holds for the non-ergodic Ornstein–Uhlenbeck model driven by Brownian motion and some explosive autoregressive models.

Remark 4.18 (Alternative parameterization)

An alternative specification of the fractional Vasicek model is

$$dX_t = \kappa\,(\mu - X_t)\,dt + \gamma dB_t^H, \quad t \in [0, T], \quad X_0 = x_0.$$

The MLEs of the parameters μ and κ have the form

$$\widehat{\mu}_T = \frac{S_T K_T - I_T J_T}{S_T J_T - w_T^H I_T}\,\gamma, \qquad \widehat{\kappa}_T = \frac{S_T J_T - w_T^H I_T}{w_T^H K_T - J_T^2}.$$

If $\kappa < 0$ and $H \in (1/2, 1)$, then

$$\begin{pmatrix} T^{1-H}\,(\widehat{\mu}_T - \mu) \\ e^{-\kappa T}\,(\widehat{\kappa}_T - \kappa) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \widetilde{\nu} \\ \eta/\widetilde{\zeta} \end{pmatrix}, \quad \text{as } T \to \infty,$$

where $\widetilde{\nu} \overset{d}{=} \mathcal{N}\left(0, \frac{\lambda_H \gamma^2}{\kappa^2}\right)$, $\eta \overset{d}{=} \mathcal{N}(0,1)$, and $\widetilde{\zeta} \overset{d}{=} \mathcal{N}\left(\frac{(x_0-\mu)\rho_H \sqrt{\lambda_H^*}(-\kappa)^{H-1}}{\sqrt{2\pi}}, \frac{1}{4\kappa^2 \sin \pi H}\right)$ are independent random variables.

The proof is carried out by delta-method.

Now let us consider the situation when one of the parameters is known. In this case we can obtain strong consistency of the corresponding MLEs (instead of weak one) by applying the strong law of large numbers for martingales.

Theorem 4.19

Let $\beta < 0$ be known and $H \in (1/2, 1)$. The MLE for α is

$$\widetilde{\alpha}_T = \frac{\gamma}{w_T^H}\left(S_T + \beta J_T\right).$$

It is unbiased, strongly consistent and normal:

$$T^{1-H}\left(\widetilde{\alpha}_T - \alpha\right) \overset{d}{=} \mathcal{N}\left(0, \lambda_H \gamma^2\right).$$

Remark 4.20

Actually, the statement of Theorem 4.19 is true regardless of the sign of β.

Theorem 4.21

Let α be known, $H \in (1/2, 1)$ and $\beta < 0$. The MLE for β is

$$\widetilde{\beta}_T = \frac{\frac{\alpha}{\gamma} J_T - I_T}{K_T}.$$

It is strongly consistent and

$$e^{-\beta T} \left(\widetilde{\beta}_T - \beta \right) \xrightarrow{d} \frac{\eta}{\zeta}, \quad \text{as } T \to \infty,$$

where η and ζ are the same as in Theorem 4.16.

Basic references for Lecture 2

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