Fractional Vasicek model: Asymptotic properties and statistical inference

Kostiantyn Ralchenko

University of Vaasa
Taras Shevchenko National University of Kyiv

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Model

In this section we study the fractional Vasicek model, described by the stochastic differential equation

$$X_t = x_0 + \int_0^t (\alpha - \beta X_s)\, ds + \gamma B_t^H, \quad t \ge 0. \tag{1}$$

We assume that the parameters $x_0 \in \mathbb{R}$, $\gamma > 0$ and $H \in (0,1)$ are known. The parameters $\alpha \in \mathbb{R}$ and $\beta > 0$ are fixed but unknown. The main goal is to estimate parameters α and β.
The equation (1) has a unique solution, which is given by

$$X_t = x_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \gamma \int_0^t e^{-\beta(t-s)} dB_s^H, \quad t \ge 0. \tag{2}$$

where $\int_0^t e^{-\beta(t-s)} dB_s^H$ is a path-wise Riemann–Stieltjes integral.

We have already proved in Section 2 that the Gaussian process

$$Y_t = \gamma \int_{-\infty}^{t} e^{-\beta(t-s)} dB_s^H$$

is stationary and ergodic. Moreover,

$$\mathrm{E}[Y_0] = 0,$$
$$\mathrm{E}\left[Y_0^2\right] = \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}}.$$

Continuous observations

Assume that we observe a trajectory of X continuously on the interval $[0, T]$.

First, we study the asymptotic behavior of the integrals $\int_0^T X_t\,dt$ and $\int_0^T X_t^2\,dt$ as $T \to \infty$. The next result gives us an idea for the construction of estimators.

Lemma 5.1

Let $H \in (0,1)$. Then

$$\frac{1}{T}\int_0^T X_t\,dt \to \frac{\alpha}{\beta}, \tag{3}$$

$$\frac{1}{T}\int_0^T X_t^2\,dt \to \frac{\alpha^2}{\beta^2} + \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}}, \tag{4}$$

a. s., as $T \to \infty$.

Proof. Let us introduce the following notation:

$$R(t) = x_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right), \quad Z_t = \gamma \int_0^t e^{-\beta(t-s)} dB_s^H.$$

Then Z_t is a *fractional Ornstein–Uhlenbeck process*, and $X_t = R(t) + Z_t$. Let us first prove the convergence (3). We have

$$\begin{aligned}\frac{1}{T}\int_0^T X_t\,dt &= \frac{1}{T}\left(x_0 - \frac{\alpha}{\beta}\right)\int_0^T e^{-\beta t}\,dt + \frac{1}{T}\int_0^T \frac{\alpha}{\beta}\,dt + \frac{1}{T}\int_0^T Z_t\,dt \\ &= \frac{1 - e^{-\beta T}}{\beta T}\left(x_0 - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta} + \frac{1}{T}\int_0^T Z_t\,dt.\end{aligned}$$

It is evident that first term converges to zero as $T \to \infty$. For all $t \geq 0$

$$Y_t = \gamma \int_{-\infty}^t e^{-\beta(t-s)} dB_s^H = Z_t + e^{-\beta t}\xi, \tag{5}$$

where $\xi = \gamma \int_{-\infty}^0 e^{\beta s}\,dB_s^H = Y_0$, and $(Y_t, t \geq 0)$ is a *stationary fractional Ornstein–Uhlenbeck process*, see Section 2.

Then the ergodic theorem implies that

$$\frac{1}{T}\int_0^T Y_t\,dt \to \mathrm{E}\left[Y_0\right],$$

a. s., as $T \to \infty$. Using the fact that $\mathrm{E}\left[Y_0\right] = 0$, we deduce

$$\frac{1}{T}\int_0^T Z_t\,dt \to 0, \tag{6}$$

a. s., as $T \to \infty$, which directly implies the convergence (3).
Now let us look at

$$\begin{aligned}\frac{1}{T}\int_0^T X_t^2\,dt &= \frac{1}{T}\int_0^T \left(R(t) + Z_t\right)^2\,dt\\ &= \frac{1}{T}\int_0^T R^2(t)\,dt + \frac{1}{T}\int_0^T Z_t^2\,dt + \frac{2}{T}\int_0^T R(t)Z_t\,dt.\end{aligned}$$

Take each term separately.

$$
\begin{aligned}
\frac{1}{T}\int_0^T R^2(t)\,dt &= \frac{1}{T}\int_0^T \left(x_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)\right)^2 dt \\
&= \frac{x_0^2}{T}\int_0^T e^{-2\beta t} dt + \frac{2x_0\alpha}{\beta T}\int_0^T e^{-\beta t} dt - \frac{2x_0\alpha}{\beta T}\int_0^T e^{-2\beta t} dt \\
&\quad + \frac{\alpha^2}{\beta^2 T}\int_0^T \left(1 - e^{-\beta t}\right)^2 dt \\
&= \frac{x_0^2\left(1 - e^{-2\beta T}\right)}{2\beta T} + \frac{2x_0\alpha\left(1 - e^{-\beta T}\right)}{\beta^2 T} - \frac{x_0\alpha\left(1 - e^{-2\beta T}\right)}{\beta^2 T} \\
&\quad + \frac{\alpha^2}{\beta^2}\left(1 - \frac{2\left(1 - e^{-\beta T}\right)}{\beta T} + \frac{1 - e^{-2\beta T}}{2\beta T}\right) \to \frac{\alpha^2}{\beta^2},
\end{aligned}
$$

as $T \to \infty$.

Applying again the ergodic theorem, we get

$$\frac{1}{T}\int_0^T Y_t^2\,dt \to \mathrm{E}\left[Y_0^2\right] = \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}},$$

whence

$$\begin{aligned}\frac{1}{T}\int_0^T Z_t^2\,dt &= \frac{1}{T}\int_0^T \left(Y_t - e^{-\beta t}\xi\right)^2\,dt \\ &= \frac{1}{T}\int_0^T Y_t^2\,dt - \frac{2\xi}{T}\int_0^T Y_t e^{-\beta t}\,dt + \frac{\xi^2}{T}\int_0^T e^{-2\beta t}\,dt \qquad (7) \\ &\to \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}},\end{aligned}$$

a. s., as $T \to \infty$.

And finally, using (6), (7), and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\left|\frac{2}{T}\int_0^T R(t)Z_t\,dt\right| &= \frac{2}{T}\left|\int_0^T\left(\left(x_0-\frac{\alpha}{\beta}\right)e^{-\beta t}+\frac{\alpha}{\beta}\right)Z_t\,dt\right|\\
&\le \frac{2}{T}\left|x_0-\frac{\alpha}{\beta}\right|\int_0^T\left|e^{-\beta t}Z_t\right|\,dt+\left|\frac{2\alpha}{\beta T}\int_0^T Z_t\,dt\right|\\
&\le \frac{2}{T}\left|x_0-\frac{\alpha}{\beta}\right|\left(\int_0^T e^{-2\beta t}\,dt\int_0^T Z_t^2\,dt\right)^{\frac{1}{2}}+\left|\frac{2\alpha}{\beta T}\int_0^T Z_t\,dt\right|\\
&= 2\left|x_0-\frac{\alpha}{\beta}\right|\left(\frac{1-e^{-2\beta T}}{2\beta T}\cdot\frac{1}{T}\int_0^T Z_t^2\,dt\right)^{\frac{1}{2}}+\left|\frac{2\alpha}{\beta T}\int_0^T Z_t\,dt\right|\to 0,
\end{aligned}$$

a. s., as $T\to\infty$. Thus, we obtain (4). □

We introduce the following estimators:

$$\widehat{\beta}_T^{(2)} = \left(\frac{1}{\gamma^2 H \Gamma(2H)\, T^2} \left(T \int_0^T X_t^2\, dt - \left(\int_0^T X_t\, dt \right)^2 \right) \right)^{-\frac{1}{2H}},$$

$$\widehat{\alpha}_T^{(2)} = \frac{\widehat{\beta}_T^{(2)}}{T} \int_0^T X_t\, dt.$$

Theorem 5.2

Let $H \in (0,1)$. Then the estimators $\widehat{\alpha}_T^{(2)}$ and $\widehat{\beta}_T^{(2)}$ are strongly consistent.

Proof of Theorem 5.2 follows from Lemma 5.1. □

Discrete observations

In applications usually the observations cannot be continuous. The estimators $\widehat{\alpha}_T^{(2)}$ and $\widehat{\beta}_T^{(2)}$ can be discretized as follows. Let $h > 0$. Assume that a trajectory of X is observed at times $t_k = kh$, $k = 0, 1, \dots, n$. Define

$$\widehat{\beta}_n^{(3)} = \left(\frac{1}{\gamma^2 H \Gamma(2H) n^2} \left(n \sum_{k=0}^{n-1} X_{kh}^2 - \left(\sum_{k=0}^{n-1} X_{kh} \right)^2 \right) \right)^{-\frac{1}{2H}},$$

$$\widehat{\alpha}_n^{(3)} = \frac{\widehat{\beta}_n^{(3)}}{n} \sum_{k=0}^{n-1} X_{kh}.$$

Theorem 5.3

Let $H \in (0,1)$. Then the estimators $\widehat{\alpha}_n^{(3)}$ and $\widehat{\beta}_n^{(3)}$ are strongly consistent.

Proof. It suffices to show that

$$\frac{1}{n}\sum_{k=0}^{n-1} X_{kh} \to \frac{\alpha}{\beta}, \tag{8}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} X_{kh}^2 \to \frac{\alpha^2}{\beta^2} + \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}} \tag{9}$$

a. s., as $n \to \infty$.

Let us prove the convergence (8).

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} X_{kh} &= \frac{1}{n}\sum_{k=0}^{n-1} R(kh) + \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}\\ &= \frac{1}{n}\sum_{k=0}^{n-1} e^{-\beta kh}\left(x_0 - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta} + \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}\\ &= \frac{1-e^{-\beta nh}}{n(1-e^{-\beta h})}\left(x_0 - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta} + \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}.\end{aligned}$$

The first term converges to zero as $n \to \infty$.

Similarly to the proof of Lemma 5.1, the ergodic theorem implies that

$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh} \to \mathbb{E}\left[Y_0\right] = 0, \tag{10}$$

a. s., as $n \to \infty$, where $(Y_t, t \geq 0)$ is the ergodic process defined by (5). Then

$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh} \to 0, \tag{11}$$

a. s., as $n \to \infty$. This implies the convergence (8).
Now consider the convergence (9). We have

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} X_{kh}^2 &= \frac{1}{n}\sum_{k=0}^{n-1} (R(kh) + Z_{kh})^2 \\ &= \frac{1}{n}\sum_{k=0}^{n-1} R^2(kh) + \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}^2 + \frac{2}{n}\sum_{k=0}^{n-1} R(kh)Z_{kh}. \end{aligned} \tag{12}$$

Take each term separately.

$$
\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1} R^2(kh) &= \frac{1}{n}\sum_{k=0}^{n}\left(x_0 e^{-\beta kh} + \frac{\alpha}{\beta}\left(1 - e^{-\beta kh}\right)\right)^2 \\
&= \frac{x_0^2}{n}\sum_{k=0}^{n-1} e^{-2\beta kh} + \frac{2x_0\alpha}{\beta n}\sum_{k=0}^{n-1} e^{-\beta kh} - \frac{2x_0\alpha}{\beta n}\sum_{k=0}^{n-1} e^{-2\beta kh} \\
&\quad + \frac{\alpha^2}{\beta^2 n}\sum_{k=0}^{n-1}\left(1 - e^{-\beta kh}\right)^2 \\
&= \frac{x_0^2\left(1 - e^{-2\beta nh}\right)}{n(1 - e^{-2\beta h})} + \frac{2x_0\alpha\left(1 - e^{-\beta nh}\right)}{\beta n(1 - e^{-\beta h})} - \frac{2x_0\alpha\left(1 - e^{-2\beta nh}\right)}{\beta n(1 - e^{-2\beta h})} \\
&\quad + \frac{\alpha^2}{\beta^2}\left(1 - \frac{2\left(1 - e^{-\beta nh}\right)}{n(1 - e^{-\beta h})} + \frac{1 - e^{-2\beta nh}}{n(1 - e^{-2\beta h})}\right) \to \frac{\alpha^2}{\beta^2},
\end{aligned}
\tag{13}
$$

as $n \to \infty$.

Applying again the ergodic theorem, we get

$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}^2 \to \mathrm{E}\left[Y_0^2\right] = \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}}.$$

Using the representation (5) and the convergence (10), we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}^2 = \frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}^2 - \frac{2\xi}{n}\sum_{k=0}^{n-1} e^{-\beta kh} Y_{kh} + \frac{\xi^2}{n}\sum_{k=0}^{n-1} e^{-2\beta kh} \to \frac{\gamma^2 H\Gamma(2H)}{\beta^{2H}}, \tag{14}$$

a. s., as $n \to \infty$.

Finally, we consider the third term in (12). By the Cauchy–Schwarz inequality, (11), and (14),

$$
\begin{aligned}
\left|\frac{2}{n}\sum_{k=0}^{n-1} R(kh)Z_{kh}\right| &= \frac{2}{n}\left|\sum_{k=0}^{n-1}\left(\left(x_0-\frac{\alpha}{\beta}\right)e^{-\beta kh}+\frac{\alpha}{\beta}\right)Z_{kh}\right| \\
&\le \frac{2}{n}\left|x_0-\frac{\alpha}{\beta}\right|\left|\sum_{k=0}^{n-1} e^{-\beta kh}Z_{kh}\right| + \left|\frac{2\alpha}{n\beta}\sum_{k=0}^{n-1} Z_{kh}\right| \\
&\le \frac{2}{n}\left|x_0-\frac{\alpha}{\beta}\right|\left(\sum_{k=0}^{n-1} e^{-2\beta kh}\sum_{k=0}^{n-1} Z_{kh}^2\right)^{\frac{1}{2}} + \left|\frac{2\alpha}{n\beta}\sum_{k=0}^{n-1} Z_{kh}\right| \\
&= 2\left|x_0-\frac{\alpha}{\beta}\right|\left(\frac{1-e^{-2\beta nh}}{n\left(1-e^{-2\beta h}\right)}\cdot\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}^2\right)^{\frac{1}{2}} + \left|\frac{2\alpha}{n\beta}\sum_{k=0}^{n-1} Z_{kh}\right|
\end{aligned}
$$

a. s., as $n\to\infty$. Combining this with (12), (13), and (14), we get (9). □

Model

Our focus in this section is on the estimation of unknown drift parameter $\theta = (\alpha, \beta) \in \mathbb{R} \times (0, \infty)$ in the fractional Vasicek model, which is described by the following stochastic differential equation:

$$X_t = x_0 + \int_0^t (\alpha + \beta X_s)\, ds + \gamma B_t^H, \quad t > 0, \quad X_0 = x_0 \in \mathbb{R}, \tag{15}$$

where $\alpha \in \mathbb{R}$, $\beta > 0$, $\gamma > 0$. Equivalently, $X = \{X_t, t \geq 0\}$ is the process given explicitly by

$$X_t = x_0 e^{\beta t} - \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right) + \gamma \int_0^t e^{\beta(t-s)} dB_s^H. \tag{16}$$

We define the estimator $\hat{\theta}_T = (\hat{\alpha}_T, \hat{\beta}_T)$ as follows

$$\hat{\alpha}_T = \frac{(X_T - x_0)\left(\int_0^T X_t^2 dt - \frac{1}{2}(X_T + x_0)\int_0^T X_t dt\right)}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}, \tag{17}$$

$$\hat{\beta}_T = \frac{(X_T - x_0)\left(\frac{1}{2}T(X_T + x_0) - \int_0^T X_t dt\right)}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}. \tag{18}$$

It is worth noting that both $\hat{\alpha}_T$ and $\hat{\beta}_T$ do not depend on the diffusion parameters $H > 0$ and $\gamma > 0$, which may be unknown.

Remark 6.1

The estimator $\hat{\theta}$ given by (17)–(18) can be thought of as the least squares estimator (LSE). Let us present its (formal) derivation. If we assume that the derivative $\dot{X}$ and the integral $\int_0^T X_t dX_t$ exist in some sense, then we can define LSE as the value of (α, β), which minimizes the following functional

$$\begin{aligned} F(\alpha, \beta) &= \int_0^T \left(\dot{X}_t - \alpha - \beta X_t\right)^2 dt \\ &= \int_0^T \dot{X}_t^2 dt + \alpha^2 T + \beta^2 \int_0^T X_t^2 dt - 2\alpha \int_0^T dX_t \\ &\quad - 2\beta \int_0^T X_t dX_t + 2\alpha\beta \int_0^T X_t dt. \end{aligned}$$

It is not hard to see that the minimum is achieved at the following point:

$$\hat{\alpha}_T = \frac{\int_0^T dX_t \int_0^T X_t^2 dt - \int_0^T X_t dt \int_0^T X_t dX_t}{T \int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}, \tag{19}$$

$$\hat{\beta}_T = \frac{T \int_0^T X_t dX_t - \int_0^T dX_t \int_0^T X_t dt}{T \int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}. \tag{20}$$

Note that these expressions do not contain the derivative $\dot{X}$, which may not exist. Moreover, if the process X is γ-Hölder continuous on $[0, T]$ with any $\gamma > \frac{1}{2}$, then the integral $\int_0^T X_t dX_t$ can be defined in Young's sense. In this case, it admits the representation

$$\int_0^T X_t dX_t = \frac{1}{2}\left(X_T^2 - X_0^2\right).$$

Substitution of these representations into the formulas (19)–(20) leads to the estimators (17)–(18).

Strong consistency

The following theorem is the first main result of this section.

Theorem 6.2

The estimator $(\hat{\alpha}_T, \hat{\beta}_T)$ *is strongly consistent, i.e.,*

$$\hat{\alpha}_T \to \alpha, \ \hat{\beta}_T \to \beta \quad \textit{a.s. when } T \to \infty.$$

In order to prove the strong consistency of the estimator $(\hat{\alpha}_T, \hat{\beta}_T)$, we need to study the asymptotic behavior of all processes and integrals involved into the right-hand sides of (17)–(18).

Our reasoning will be based substantially on the almost sure upper bound:

$$\sup_{0 \le s \le t} \left| B_s^H \right| \le \left(\left(t^H (\log^+ t)^p \right) \vee 1 \right) \xi, \quad \text{a.s.}, \tag{21}$$

for all $t > 0$.
Therefore,

$$\int_0^\infty e^{-\beta s} \left| B_s^H \right| ds < \infty \quad \text{a.s.},$$

and the following random variables are well-defined:

$$Z_\infty := \int_0^\infty e^{-\beta s} B_s^H \, ds, \quad \zeta := x_0 + \frac{\alpha}{\beta} + \gamma \beta Z_\infty. \tag{22}$$

Lemma 6.3

The following convergences hold a.s. when $T \to \infty$:

$$e^{-\beta T} X_T \to \zeta, \tag{23}$$

$$e^{-\beta T} \int_0^T X_t dt \to \frac{1}{\beta}\zeta, \tag{24}$$

$$e^{-\beta T} \int_0^T |X_t| \, dt \to \frac{1}{\beta} |\zeta| \, , \tag{25}$$

$$e^{-2\beta T} \int_0^T X_t^2 dt \to \frac{1}{2\beta}\zeta^2, \tag{26}$$

$$T^{-1} e^{-\beta T} \int_0^T X_t t \, dt \to \frac{1}{\beta}\zeta. \tag{27}$$

Proof. Recall that according to (16), the process X can be represented as follows:

$$X_t = x_0 e^{\beta t} - \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right) + \gamma e^{\beta t} U_t, \tag{28}$$

where

$$U_t = \int_0^t e^{-\beta s}\, dB_s^H.$$

Using the integration-by-parts formula, we can write

$$U_T = e^{-\beta T} B_T^H + \beta \int_0^T e^{-\beta s} B_s^H\, ds = e^{-\beta T} B_t^H + \beta Z_T, \tag{29}$$

where

$$Z_T = \int_0^T e^{-\beta s} B_s^H\, ds.$$

It follows from the upper bound (21) that

$$e^{-\beta T} B_T^H \to 0 \quad \text{a.s. when } T \to \infty. \tag{30}$$

Furthermore, by the dominated convergence theorem, we see that

$$Z_T \to Z_\infty \quad \text{a.s. when } T \to \infty, \tag{31}$$

and also note that all these random variables are Gaussian. Consequently, (29)–(31) imply that

$$U_T \to \beta Z_\infty \quad \text{a.s. when } T \to \infty.$$

Taking into account the representation (28), we obtain the following passage to the limit:

$$e^{-\beta T} X_T \to x_0 + \frac{\alpha}{\beta} + \gamma\beta Z_\infty = \zeta \quad \text{a.s. when } T \to \infty.$$

Thus, (23) is proved. Moreover, using l'Hôpital's rule, we immediately obtain the convergences (24)–(27) from (23). □

Lemma 6.4

The following convergence holds:

$$T^{-1}e^{-\beta T}\left(\int_0^T X_t^2 dt - \frac{1}{2}(X_T + x_0)\int_0^T X_t dt\right) \to \frac{\alpha}{2\beta}\zeta \quad a.s.\ as\ T \to \infty. \tag{32}$$

Proof. Using the stochastic differential equation (15), we can write

$$\begin{aligned}
&\int_0^T X_t^2 dt - \frac{1}{2}(X_T + x_0)\int_0^T X_t dt \\
&\quad = \int_0^T X_t\left(x_0 + \alpha t + \beta\int_0^t X_s ds + \gamma B_t^H\right) dt \\
&\quad\quad - \frac{1}{2}\left(2x_0 + \alpha T + \beta\int_0^T X_t dt + \gamma B_T^H\right)\int_0^T X_t dt
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \left(\int_0^T X_t t\, dt - \frac{1}{2} T \int_0^T X_t dt \right) \\
&\quad + \beta \left(\int_0^T X_t \int_0^t X_s\, ds\, dt - \frac{1}{2} \left(\int_0^T X_t dt \right)^2 \right) \\
&\quad + \gamma \left(\int_0^T X_t B_t^H\, dt - \frac{1}{2} B_T^H \int_0^T X_t dt \right) \\
&=: \alpha I_\alpha(T) + \beta I_\beta(T) + \gamma I_\gamma(T).
\end{aligned}
$$

It is not hard to see that $\int_0^T X_t \int_0^t X_s\, ds\, dt = \frac{1}{2} \left(\int_0^T X_t dt \right)^2$, so $I_\beta(T) \equiv 0$. Furthermore, using the convergences (24) and (27), we obtain that

$$
\begin{aligned}
T^{-1} e^{-\beta T} I_\alpha(T) &= T^{-1} e^{-\beta T} \int_0^T X_t t\, dt - \frac{1}{2} e^{-\beta T} \int_0^T X_t dt \\
&\to \frac{1}{\beta}\zeta - \frac{1}{2\beta}\zeta = \frac{1}{2\beta}\zeta \quad \text{a.s. when } T \to \infty.
\end{aligned}
$$

Now it remains to prove that $T^{-1}e^{-\beta T}I_\gamma(T)$ tends to zero a.s. when $T \to \infty$. Note that according to (21),

$$\frac{1}{T}\sup_{0\le t\le T}\left|B_t^H\right| \to 0 \quad \text{a.s. when } T \to \infty. \tag{33}$$

Therefore,

$$\begin{aligned} &T^{-1}e^{-\beta T}\left|I_\gamma(T)\right| \\ &\quad\le T^{-1}e^{-\beta T}\int_0^T \left|X_t B_t^H\right| dt + \frac{1}{2}T^{-1}e^{-\beta T}\left|B_T^H\right|\int_0^T |X_t|\,dt \\ &\quad\le \frac{3}{2}\cdot\frac{1}{T}\sup_{0\le t\le T}\left|B_t^H\right| e^{-\beta T}\int_0^T |X_t|\,dt \to 0 \quad \text{a.s. as } T\to\infty, \end{aligned}$$

by (33) and (25). □

Proof of Theorem 6.2

In order to prove the strong consistency of $\hat{\beta}_T$, we rewrite it in the following form:

$$\hat{\beta}_T = \frac{\frac{1}{2}e^{-2\beta T}(X_T^2 - x_0^2) - e^{-\beta T}(X_T - x_0)T^{-1}e^{-\beta T}\int_0^T X_t dt}{e^{-2\beta T}\int_0^T X_t^2 dt - \left(T^{-1/2}e^{-\beta T}\int_0^T X_t dt\right)^2} =: \frac{B(T)}{D(T)}. \tag{34}$$

It follows from (23) and (24) that only the term $\frac{1}{2}e^{-2\beta T}X_T^2$ in the numerator converges a.s. to the non-zero limit. Namely,

$$B(T) \to \frac{1}{2}\zeta^2 \quad \text{a.s. when } T \to \infty.$$

Further, we obtain from (24) and (26):

$$D(T) \to \frac{1}{2\beta}\zeta^2 \quad \text{a.s. when } T \to \infty. \tag{35}$$

Thus, $\hat{\beta}_T \to \beta$ a.s. when $T \to \infty$.

Now, let us consider $\hat{\alpha}_T$.

$$\hat{\alpha}_T = \frac{e^{-\beta T}(X_T - x_0) T^{-1} e^{-\beta T} \left(\int_0^T X_t^2 dt - \frac{1}{2}(X_T + x_0) \int_0^T X_t dt \right)}{e^{-2\beta T} \int_0^T X_t^2 dt - \left(T^{-1/2} e^{-\beta T} \int_0^T X_t dt \right)^2} =: \frac{A(T)}{D(T)} \tag{36}$$

It follows from (23) and (32) that

$$A(T) \to \frac{\alpha}{2\beta} \zeta^2 \quad \text{a.s. when } T \to \infty.$$

Combining this convergence with (36) and (35), we obtain that $\hat{\alpha}_T \to \alpha$ a.s. when $T \to \infty$. □

Asymptotic distribution

Theorem 6.5

Let

$$\sigma^2_{H,\beta} = \frac{H\Gamma(2H)}{\beta^{2H}}. \tag{37}$$

Then

$$\begin{pmatrix} T^{1-H}\left(\hat{\alpha}_T - \alpha\right) \\ e^{\beta T}\left(\hat{\beta}_T - \beta\right) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \eta_1 \\ \frac{\eta_3}{\eta_2} \end{pmatrix} \quad \text{as } T \to \infty,$$

where $\eta_1 \simeq \mathcal{N}(0, \gamma^2)$, $\eta_2 \simeq \mathcal{N}(x_0 + \frac{\alpha}{\beta}, \gamma^2\sigma^2_{H,\beta})$ *and* $\eta_3 \simeq \mathcal{N}(0, 4\beta^2\gamma^2\sigma^2_{H,\beta})$ *are jointly independent random variables.*

Proof

Let us introduce the following processes

$$Z_t := \int_0^t e^{-\beta s} B_s^H ds, \tag{38}$$

$$V_t := e^{-\beta t} \int_0^t e^{\beta s} dB_s^H = B_t^H - \beta e^{-\beta t} \int_0^t e^{\beta s} B_s^H ds. \tag{39}$$

The proof of Theorem 6.5 follows the following scheme.

- First, we express $T^{1-H}(\hat{\alpha}_T - \alpha)$ and $e^{\beta T}(\hat{\beta}_T - \beta)$ via the processes B^H, Z, V, and remainder terms, vanishing at infinity.
- Then we find the joint asymptotic distribution of the Gaussian vector $(T^{-H} B_T^H, Z_T, V_T)$ as $T \to \infty$.
- Finally, using these results along with the Slutsky theorem, we derive the limits in distribution for $T^{1-H}(\hat{\alpha}_T - \alpha)$ and $e^{\beta T}(\hat{\beta}_T - \beta)$ as $T \to \infty$.

Representation of the estimators

Lemma 6.6

For all $T > 0$

$$e^{\beta T}(\hat{\beta}_T - \beta) = \frac{\gamma V_T \left(x_0 + \frac{\alpha}{\beta} + \beta\gamma Z_T \right)}{D_T} + R_T, \tag{40}$$

where

$$D_T := e^{-2\beta T}\left(\int_0^T X_t^2 dt - \frac{1}{T}\left(\int_0^T X_t dt \right)^2 \right) \to \frac{1}{2\beta}\zeta^2 \quad a.s., \text{ as } T \to \infty, \tag{41}$$

and

$$R_T \to 0 \quad a.s., \text{ as } T \to \infty. \tag{42}$$

Lemma 6.7

For all $T > 0$

$$T^{1-H}(\hat{\alpha}_T - \alpha) = \frac{\gamma B_T^H}{T^H} + Q_T,$$

where

$$Q_T = -\left(\hat{\beta}_T - \beta\right) \frac{1}{T^H} \int_0^T X_t dt. \tag{43}$$

Proof. Using (17) and (18) we rewrite $T\hat{\alpha}_T$ as follows

$$T\hat{\alpha}_T = X_T - x_0 - \hat{\beta}_T \int_0^T X_t dt.$$

Now expressing X_T through (15), we get

$$T\hat{\alpha}_T = T\alpha + \gamma B_T^H - \left(\hat{\beta}_T - \beta\right) \int_0^T X_t dt.$$

whence the proof follows. □

Asymptotic normality of $(T^{-H}B_T^H, Z_T, V_T)$

Proposition 6.8

As $T \to \infty$,

$$\begin{pmatrix} T^{-H}B_T^H \\ Z_T \\ V_T \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma),$$

where

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sigma_{H,\beta}^2}{\beta^2} & 0 \\ 0 & 0 & \sigma_{H,\beta}^2 \end{pmatrix}.$$

In particular, B_T^H, Z_T, and V_T are asymptotically independent.

Proof. In view of the normality of the random vector $(T^{-H}B_T^H, Z_T, V_T)$, it suffices to find the elements of its asymptotic covariance matrix.
(a) We know that

$$\mathrm{E}\left[\left(T^{-H}B_T^H\right)^2\right] = 1.$$

(b) The following convergence is a corollary of Lemma 3.1:

$$\lim_{T\to\infty} \mathrm{E}Z_T^2 = \mathrm{E}Z_\infty^2 = \frac{\sigma_{H,\beta}^2}{\beta^2} \tag{44}$$

(c) Note that $V_t = e^{-\beta t}\int_0^t e^{\beta s} dB_s^H$ is a fractional Ornstein–Uhlenbeck process with $\theta = -\beta$ and $\sigma = 1$, whence

$$\lim_{T\to\infty} \mathrm{E}V_T^2 = \lim_{T\to\infty} v(-\beta, T) = \sigma_{H,\beta}^2, \tag{45}$$

by Lemma 2.3.

(d) $\mathrm{E}\left[T^{-H}B_T^H Z_T\right] = T^{-H}\mathrm{E}\left[B_T^H \int_0^T e^{-\beta t} B_t^H dt\right]$

$$= \frac{1}{2} T^{-H} \int_0^T e^{-\beta t}\left(T^{2H} + t^{2H} - |T-t|^{2H}\right) dt$$

$$= \frac{1}{2} T^{H} \cdot \frac{1-e^{-\beta T}}{\beta} + \frac{1}{2} T^{-H} \int_0^T e^{-\beta t} t^{2H} dt - \frac{1}{2} T^{-H} \int_0^T e^{-\beta(T-s)} s^{2H} ds.$$

By integration by parts, the last term equals

$$\frac{1}{2} T^{-H} \int_0^T e^{-\beta(T-s)} s^{2H} ds = \frac{T^H}{2\beta} - \frac{H}{\beta} T^{-H} \int_0^T e^{-\beta(T-s)} s^{2H-1} ds.$$

Hence,

$$\mathrm{E}\left[T^{-H}B_T^H Z_T\right] = -\frac{T^H e^{-\beta T}}{2\beta} + \frac{1}{2T^H} \int_0^T e^{-\beta t} t^{2H} dt + \frac{H}{\beta T^H} \int_0^T e^{-\beta(T-s)} s^{2H-1} ds \to 0 \quad \text{as } T \to \infty.$$

(e) Similarly one can prove that

$$
\begin{aligned}
\mathrm{E}\left[T^{-H}B_T^H V_T\right] &= T^{-H}e^{-\beta T}\mathrm{E}\left[(B_T^H)^2 - \beta B_T^H\int_0^T e^{\beta t}B_t^H dt\right] \\
&= T^{2H} - \frac{\beta}{2}T^{-H}e^{-\beta T}\int_0^T e^{\beta t}\left(T^{2H} + T^{2H} - |T-t|^{2H}\right)dt \\
&= T^{2H} - \frac{T^H}{2} + \frac{T^H}{2e^{\beta T}} - \frac{\beta}{2}T^{-H}e^{-\beta T}\int_0^T e^{\beta t}t^{2H}dt \\
&\quad + \frac{\beta}{2}T^{-H}\int_0^T e^{-\beta s}s^{2H}ds \\
&= \frac{T^H}{2e^{\beta T}} + HT^{-H}\int_0^T e^{-\beta(T-s)}s^{2H-1}ds + \frac{\beta}{2}T^{-H}\int_0^T e^{-\beta s}s^{2H}ds \\
&\to 0 \quad \text{as } T\to\infty.
\end{aligned}
$$

(f) The term $\mathrm{E}\left[Z_T V_T\right]$ is analyzed in a similar manner. We get

$$\lim_{T\to\infty}\mathrm{E}\left[Z_T V_T\right] = 0.$$

The proof is complete.

Proof of Theorem 6.5

Let $(\xi_1, \xi_2, \xi_3) \simeq \mathcal{N}(\mathbf{0}, \Sigma)$, where the matrix Σ is defined in Prop. 6.8.
(i) According to Lemma 6.6, one has the following representation

$$e^{\beta t}\left(\hat{\beta}_T - \beta\right) = \frac{\left(x_0 + \frac{\alpha}{\beta} + \beta\gamma Z_T\right)^2}{D_T} \frac{\gamma V_T}{x_0 + \frac{\alpha}{\beta} + \beta\gamma Z_T} + R_T,$$

where $R_T \to 0$ a.s. when $T \to \infty$. By (22) and (41),

$$\frac{\left(x_0 + \frac{\alpha}{\beta} + \beta\gamma Z_T\right)^2}{D_T} \to \frac{\zeta^2}{\frac{1}{2\beta}\zeta^2} = 2\beta \quad \text{a.s., as } T \to \infty.$$

Therefore, we derive from Proposition 6.8 and the Slutsky theorem that

$$e^{\beta t}\left(\hat{\beta}_T - \beta\right) \xrightarrow{d} \frac{2\beta\gamma\xi_3}{x_0 + \frac{\alpha}{\beta} + \beta\gamma\xi_2} = \frac{\eta_3}{\eta_2},$$

where $\eta_3 := 2\beta\gamma\xi_3 \simeq \mathcal{N}(0, 4\beta^2\gamma^2\sigma^2_{H,\beta})$ and
$\eta_2 := x_0 + \frac{\alpha}{\beta} + \beta\gamma\xi_2 \simeq \mathcal{N}(x_0 + \frac{\alpha}{\beta}, \gamma^2\sigma^2_{H,\beta})$ are uncorrelated, hence, independent.

By Lemma 6.7 and Proposition 6.8, For all $T > 0$

$$T^{1-H}(\hat{\alpha}_T - \alpha) = \frac{\gamma B_T^H}{T^H} + Q_T,$$

where

$$Q_T = -e^{\beta T}\left(\hat{\beta}_T - \beta\right)\frac{e^{-\beta T}}{T^H}\int_0^T X_t dt \xrightarrow{\mathrm{P}} 0.$$

Proposition 6.8 and the Slutsky theorem,

$$T^{1-H}(\hat{\alpha}_T - \alpha) \xrightarrow{d} \gamma\xi_1 =: \eta_1 \simeq \mathcal{N}(0, \gamma^2),$$

and η_1 is independent of η_2 and η_3. □

Mandelbrodt–Van Ness representation of fBm

According to the Mandelbrot–Van Ness representation, or moving-average representation, a fractional Brownian motion B^H can be represented as

$$B_t^H = \int_{-\infty}^{t} z(t,s)\, dW_s,$$

where W is a two-sided Wiener process, and the Volterra kernel z is defined by the formula

$$z(t,s) = c_H \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right),$$

$$c_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{1/2} = \frac{(2H\sin(\pi H)\Gamma(2H))^{1/2}}{\Gamma(H+\frac{1}{2})}, \quad x_+ = \max\{x, 0\}.$$

B. B. Mandelbrot and J. W. Van Ness.
Fractional Brownian motions, fractional noises and applications.
SIAM Rev., 10:422–437, 1968.

Tempered fractional Brownian motion

Let $W = \{W_s,\ s \in \mathbb{R}\}$ be a two-sided Wiener process, $H > 0$, $\lambda > 0$.

Definition 7.1 ([Meerschaert and Sabzikar (2013)])

The stochastic process $B_{H,\lambda} = \{B_{H,\lambda}(t)\}_{t\in\mathbb{R}}$ defined by the Wiener integral

$$B_{H,\lambda}(t) := \int_{-\infty}^{t} g_{H,\lambda}(t,s)dW_s, \tag{46}$$

where

$$g_{H,\lambda}(t,s) := e^{-\lambda(t-s)_+}(t-s)_+^{H-\frac{1}{2}} - e^{-\lambda(-s)_+}(-s)_+^{H-\frac{1}{2}}, \quad s \in \mathbb{R},$$

is called a tempered fractional Brownian motion (TFBM).

Mark M. Meerschaert and Farzad Sabzikar.
Tempered fractional Brownian motion.
Stat. Probab. Lett., 83(10):2269–2275, 2013.

Basic properties of TFBM

1. TFBM (46) is a Gaussian stochastic process with stationary increments, having the following scaling property: for any scaling factor $c > 0$
$$\{B_{H,\lambda}(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{c^H B_{H,c\lambda}(t)\}_{t\in\mathbb{R}}, \tag{47}$$
(here $\stackrel{d}{=}$ denotes the equality of all finite-dimensional distributions).

2. From (47)

$$\mathrm{E}\big[(B_{H,\lambda}(|t|))^2\big] = |t|^{2H}\mathrm{E}\big[(B_{H,\lambda|t|}(1))^2\big] := |t|^{2H} C_t^2, \qquad (48)$$

where the function C_t^2 has the following explicit representation.

$$C_t^2 = (C_t)^2 = \mathrm{E}\big[(B_{H,\lambda|t|}(1))^2\big] = \frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}} - \frac{2\Gamma(H+\frac{1}{2})}{\sqrt{\pi}} \frac{1}{(2\lambda|t|)^{H}} K_H(\lambda|t|),$$

where $t \neq 0$ and $K_\nu(z)$ is the modified Bessel function of the second kind.

2. From (47)

$$\mathrm{E}\big[(B_{H,\lambda}(|t|))^2\big] = |t|^{2H}\mathrm{E}\big[(B_{H,\lambda|t|}(1))^2\big] := |t|^{2H} C_t^2, \qquad (48)$$

where the function C_t^2 has the following explicit representation.

$$C_t^2 = (C_t)^2 = \mathrm{E}\big[(B_{H,\lambda|t|}(1))^2\big] = \frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}} - \frac{2\Gamma(H+\frac{1}{2})}{\sqrt{\pi}} \frac{1}{(2\lambda|t|)^H} K_H(\lambda|t|),$$

where $t \neq 0$ and $K_\nu(z)$ is the modified Bessel function of the second kind.

3. TFBM (46) with parameters $H > 0$ and $\lambda > 0$ has the covariance function

$$\operatorname{cov}[B_{H,\lambda}(t), B_{H,\lambda}(s)] = \frac{1}{2}\Big[C_t^2|t|^{2H} + C_s^2|s|^{2H} - C_{t-s}^2|t-s|^{2H}\Big],$$

for any $t, s \in \mathbb{R}$.

Proposition 7.2 ([Azmoodeh, Mishura, and Sabzikar (2022)])

1. *The TFBM* (46) *with parameters* $H > 0$ *and* $\lambda > 0$ *satisfies*

$$\lim_{t \to +\infty} \mathrm{E}\left[B_{H,\lambda}(t)\right]^2 = \frac{2\Gamma(2H)}{(2\lambda)^{2H}}. \tag{49}$$

2. *Let* $B_{H,\lambda}$ *be a TFBM from* (46) *with* $0 < H < 1$ *and* $\lambda > 0$. *Then, there exist positive constants* C_1 *and* C_2 *such that for any* $s, t \in [0, 1]$

$$C_1 |t - s|^{2H} \le \mathrm{E}[|B_{H,\lambda}(t) - B_{H,\lambda}(s)|^2] \le C_2 |t - s|^{2H}, \tag{50}$$

Ehsan Azmoodeh, Yuliya Mishura, and Farzad Sabzikar.
How does tempering affect the local and global properties of fractional Brownian motion?
J. Theor. Probab., 35(1):484–527, 2022.

Asymptotic behavior of variance of TFBM at zero

Lemma 7.3

1. *For any* $H \in (0,1)$

$$\mathrm{E}\left[B_{H,\lambda}(t)^2\right] \sim \frac{\Gamma(H+\frac{1}{2})^2}{2H\sin(\pi H)\Gamma(2H)}\, t^{2H}, \quad \textit{as } t \downarrow 0. \tag{51}$$

2. *For* $H = 1$

$$\mathrm{E}\left[B_{1,\lambda}(t)^2\right] \sim -\frac{t^2}{4}\log t, \quad \textit{as } t \downarrow 0.$$

3. *For any* $H > 1$

$$\mathrm{E}\left[B_{H,\lambda}(t)^2\right] \sim \frac{\Gamma(2H)}{2^{2H+1}\lambda^{2H-2}(H-1)}\, t^2, \quad \textit{as } t \downarrow 0.$$

Remark 7.4

Let $H \in (0,1)$. When $\lambda = 0$, we get (up to multiplicative constant) the conventional (untempered) fractional Brownian motion (FBM) through the Mandelbrodt–van Ness representation:

$$B_H(t) := \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right] dW_s. \tag{52}$$

It is known that this process has the variance

$$\mathrm{E}\left[(B_H(t))^2 \right] = \frac{\Gamma(H+\frac{1}{2})^2}{2H\sin(\pi H)\Gamma(2H)} \cdot t^{2H}, \quad t \geq 0.$$

Hence, the relation (51) means that the variance of TFBM is asymptotically equivalent to the variance of FBM as $t \downarrow 0$.
Note that in the case $H \geq 1$ the integral in (52) does not exists. The FBM is defined for $H = 1$ by different approach. This explains the difference in the behavior of TFBM and FBM in the case $H = 1$.

Lemma 7.5

1. *For any* $H \in (0,1)$ *and for all* $t, s \in \mathbb{R}_+$

$$\mathrm{E}\left[\left|B_{H,\lambda}(t) - B_{H,\lambda}(s)\right|^2\right] \le C\left(|t-s|^{2H} \wedge 1\right).$$

2. *If* $H = 1$, *then for all* $t, s \in \mathbb{R}_+$

$$\mathrm{E}\left[\left|B_{H,\lambda}(t) - B_{H,\lambda}(s)\right|^2\right] \le C\left(|t-s|^2 \left|\log|t-s|\right| \wedge 1\right).$$

3. *For any* $H > 1$ *and for all* $t, s \in \mathbb{R}_+$

$$\mathrm{E}\left[\left|B_{H,\lambda}(t) - B_{H,\lambda}(s)\right|^2\right] \le C(|t-s|^2 \wedge 1).$$

Remark 7.6 (sample path properties of TFBM)

[Meerschaert and Sabzikar (2013)] have obtained the following results about sample path properties of TFBM:

(a) If $H \in (0, 1)$, then TFBM satisfies Hölder condition of any order $\gamma \in (0, H)$ on any interval $[0, T]$;

(b) For $H > 1$ TFBM has continuously differentiable sample paths and, moreover, they are p times continuously differentiable for $H > p$.

Note that (a) agrees with statement (i) of our Lemma 7.5, and follows from it via the Kolmogorov–Čentsov theorem.

(c) for $H = 1$, the process $B_{1,\lambda}(t)$ satisfies Hölder condition of any order $\gamma \in (0, 1)$; it is not continuously differentiable for any $t \geq 0$.

Asymptotic growth of the trajectories of TFBM

Theorem 7.7

For any $\delta > 0$ there exists a non-negative random variable $\xi = \xi(\delta)$ such that for all $t > 0$

$$\sup_{s\in[0,t]} |B_{H,\lambda}(s)| \le \left(t^\delta \vee 1\right)\xi \quad a.s.,$$

and there exist positive constants $C_1 = C_1(\delta)$ and $C_2 = C_2(\delta)$ such that for all $u > 0$

$$\mathrm{P}(\xi > u) \le C_1 e^{-C_2 u^2}. \tag{53}$$

Let $\alpha = \frac{K_1}{1-e^{-\delta}}$. Then for any $\gamma > 0$ there exists a constant $C_3 > 0$ such that for all $u > \alpha$

$$\mathrm{P}(\xi > u) \le 2^{\frac{2}{\beta}-1}\sqrt{e}\exp\left\{-\frac{u^2}{2\alpha^2}\right\}\exp\left\{C_3\left(1-\sqrt{1-\frac{\alpha^2}{u^2}}\right)^{-\gamma/\beta}\right\}.$$

Remark 7.8

For FBM $B^H = \{B_t^H, t \geq 0\}$, the following upper bound was established in [Kozachenko, Melnikov, and Mishura (2015)]

$$\sup_{s\in[0,t]} \left|B_s^H\right| \leq \left(1 \vee \left(t^H (\log^+ t)^p\right)\right) \xi(p) \tag{54}$$

for any $p > 0$. We see that TBFM demonstrates much slower asymptotic growth.

Yuriy Kozachenko, Alexander Melnikov, and Yuliya Mishura.
On drift parameter estimation in models with fractional Brownian motion.
Statistics, 49(1):35–62, 2015.

Strongly consistent drift estimation in the fractional tempered Vasicek model

Our focus in this section is on the statistical inference of the tempered fractional Vasicek model, which is described by the following stochastic differential equation:

$$X_t = x_0 + \int_0^t (\alpha + \beta X_s)\, ds + \gamma B_{H,\lambda}(t), \quad t > 0, \quad X_0 = x_0 \in \mathbb{R}, \tag{55}$$

where $\alpha \in \mathbb{R}$, $\beta > 0$, $\gamma > 0$. Equivalently, $X = \{X_t, t \geq 0\}$ is the process given explicitly by

$$X_t = x_0 e^{\beta t} - \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right) + \gamma \int_0^t e^{\beta(t-s)} dB_{H,\lambda}(s). \tag{56}$$

Let us consider the estimation of unknown drift parameter $\theta = (\alpha, \beta) \in \mathbb{R} \times (0, \infty)$ in the Vasicek model (55).

We define the estimator $\hat{\theta}_T = (\hat{\alpha}_T, \hat{\beta}_T)$ as follows

$$\hat{\alpha}_T = \frac{(X_T - x_0)\left(\int_0^T X_t^2 dt - \frac{1}{2}(X_T + x_0)\int_0^T X_t dt\right)}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}, \tag{57}$$

$$\hat{\beta}_T = \frac{(X_T - x_0)\left(\frac{1}{2}T(X_T + x_0) - \int_0^T X_t dt\right)}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}. \tag{58}$$

It is worth noting that both $\hat{\alpha}_T$ and $\hat{\beta}_T$ do not depend on the diffusion parameters $H > 0$, $\lambda > 0$, and $\gamma > 0$, which may be unknown.

Theorem 7.9

The estimator $(\hat{\alpha}_T, \hat{\beta}_T)$ is strongly consistent, i.e.,

$$\hat{\alpha}_T \to \alpha, \ \hat{\beta}_T \to \beta \quad \textit{a.s. when } T \to \infty.$$

The strong consistency of the estimator $(\hat{\alpha}_T, \hat{\beta}_T)$, is proved similarly to the fractional Brownian case. We study the asymptotic behavior of all processes and integrals involved into the right-hand sides of (57)–(58), using the asymptotic growth of TFBM's sample path (Theorem 7.7). As in the case of the fractional Vasicek model,

$$\int_0^\infty e^{-\beta s} \left| B_{H,\lambda}(s) \right| ds < \infty \quad \text{a.s.},$$

and the following random variable is well defined:

$$\zeta := x_0 + \frac{\alpha}{\beta} + \gamma\beta \int_0^\infty e^{-\beta s} B_{H,\lambda}(s)\, ds.$$

Lemma 7.10

The following convergences hold a.s. when $T \to \infty$:

$$e^{-\beta T} X_T \to \zeta,$$

$$e^{-\beta T} \int_0^T X_t dt \to \frac{1}{\beta}\zeta, \quad e^{-\beta T} \int_0^T |X_t| \, dt \to \frac{1}{\beta} |\zeta| \, ,$$

$$e^{-2\beta T} \int_0^T X_t^2 dt \to \frac{1}{2\beta}\zeta^2, \quad T^{-1} e^{-\beta T} \int_0^T X_t t \, dt \to \frac{1}{\beta}\zeta,$$

$$T^{-1} e^{-\beta T} \left(\int_0^T X_t^2 dt - \frac{1}{2}(X_T + x_0) \int_0^T X_t dt \right) \to \frac{\alpha}{2\beta}\zeta.$$

Using the above convergences, we establish strong consistency similarly to the the case of the non-ergodic fractional Vasicek model studied in Section 6.

Asymptotic distribution of the estimator

Theorem 7.11

Let

$$\varrho^2_{H,\lambda} = \frac{2\Gamma(2H)}{(2\lambda)^{2H}} \quad \text{and} \quad \sigma^2_{H,\lambda,\beta} = \frac{\beta}{2}\int_0^\infty \exp\{-\beta u\} C_u^2 u^{2H} du. \tag{59}$$

Then the following statements hold.

1. *The estimator $\hat{\alpha}_T$ is asymptotically normal:*

$$T\left(\hat{\alpha}_T - \alpha\right) \xrightarrow{d} \mathcal{N}\left(0, \gamma^2 \varrho^2_{H,\lambda}\right) \quad \text{as } T \to \infty.$$

2. *The estimator $\hat{\beta}_T$ has asymptotic Cauchy-type distribution:*

$$e^{\beta T}\left(\hat{\beta}_T - \beta\right) \xrightarrow{d} \frac{\eta_1}{\eta_2}, \quad \text{as } T \to \infty,$$

where $\eta_1 \simeq \mathcal{N}(0, 4\beta^2\gamma^2\sigma^2_{H,\lambda,\beta})$ and $\eta_2 \simeq \mathcal{N}(x_0 + \frac{\alpha}{\beta}, \gamma^2\sigma^2_{H,\lambda,\beta})$ are independent normal random variables.

$$Z_T := \int_0^T e^{-\beta s} B_{H,\lambda}(s)ds, \quad U_T = e^{-\beta T} \int_0^T e^{\beta s} B_{H,\lambda}(s)ds,$$

$$V_T := e^{-\beta T} \int_0^T e^{\beta s} dB_{H,\lambda}(s) = B_{H,\lambda}(T) - \beta U_T.$$

Proposition 7.12

$$\begin{pmatrix} Z_T \\ U_T \\ V_T \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma), \quad \text{as } T \to \infty$$

where

$$\Sigma = \begin{pmatrix} \beta^{-2}\sigma^2_{H,\lambda,b} & \beta^{-2}\sigma^2_{H,\lambda,b} & 0 \\ \beta^{-2}\sigma^2_{H,\lambda,b} & \beta^{-2}\left(\varrho^2_{H,\lambda} - \sigma^2_{H,\lambda,b}\right) & 0 \\ 0 & 0 & \sigma^2_{H,\lambda,b} \end{pmatrix}.$$

In particular,

- *Z_T and V_T are asymptotically independent;*
- *U_T and V_T are asymptotically independent.*

Remark 7.13 (Joint distribution of the estimators)

Unlike the case of the Vasicek model driven by fractional Brownian motion, the estimators $\hat{\alpha}_T$ and $\hat{\beta}_T$ are *not* asymptotically independent. More precisely, the following convergence holds

$$\begin{pmatrix} T\left(\hat{\alpha}_T - \alpha\right) \\ e^{\beta T}\left(\hat{\beta}_T - \beta\right) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \beta\gamma\xi_2 - \gamma\xi_3 \\ \frac{2\beta\gamma\xi_3}{x_0 + \frac{\alpha}{\beta} + \beta\gamma\xi_1} \end{pmatrix}, \tag{60}$$

where the random vector (ξ_1, ξ_2, ξ_3) has a Gaussian distribution $\mathcal{N}(\mathbf{0}, \Sigma)$ with the covariance matrix Σ defined in Proposition 7.12. We see that the normal random variables $\xi_1 \simeq \mathcal{N}(0, \beta^{-2}\sigma^2_{H,\lambda,\beta})$ and $\xi_3 \simeq \mathcal{N}(0, \sigma^2_{H,\lambda,\beta})$ are independent, and so are $\xi_2 \simeq \mathcal{N}(0, \beta^{-2}(\varrho^2_{H,\lambda} - \sigma^2_{H,\lambda,\beta}))$ and ξ_3. However, there is a correlation between ξ_1 and ξ_2, namely $\operatorname{cov}(\xi_1, \xi_2) = \beta^{-2}\sigma^2_{H,\lambda,\beta}$.

- Stanislav Lohvinenko, Kostiantyn Ralchenko, and Olha Zhuchenko.
  Asymptotic properties of parameter estimators in fractional Vasicek model.
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