Free Probability Theory
and
Random Matrices

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Free Probability Theory

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Some History of Free Probability

1985 Voiculescu introduces ”freeness” in the context of operator algebras (isomorphism problem of free group factors)

1990 Combinatorial theory of freeness, based on ”free cumulants” Nica, Speicher: *Lectures on the combinatoris of free probability*

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$\longrightarrow$ use of free probability in statistical physics, wireless networks, machine learning ...

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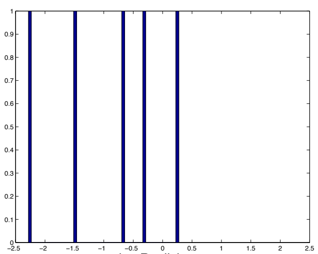
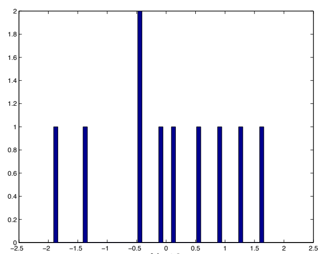
Typical phenomena for basic random matrix ensembles:

- almost sure convergence to a deterministic limit eigenvalue distribution

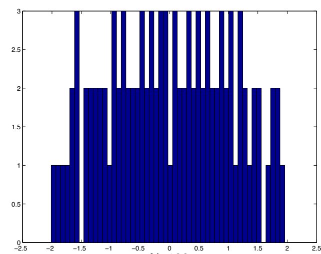
- this limit distribution can be effectively calculated



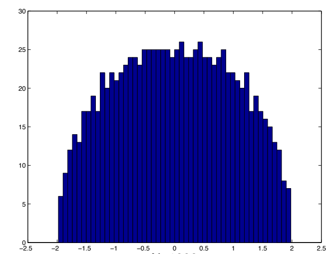
eine Realisierung



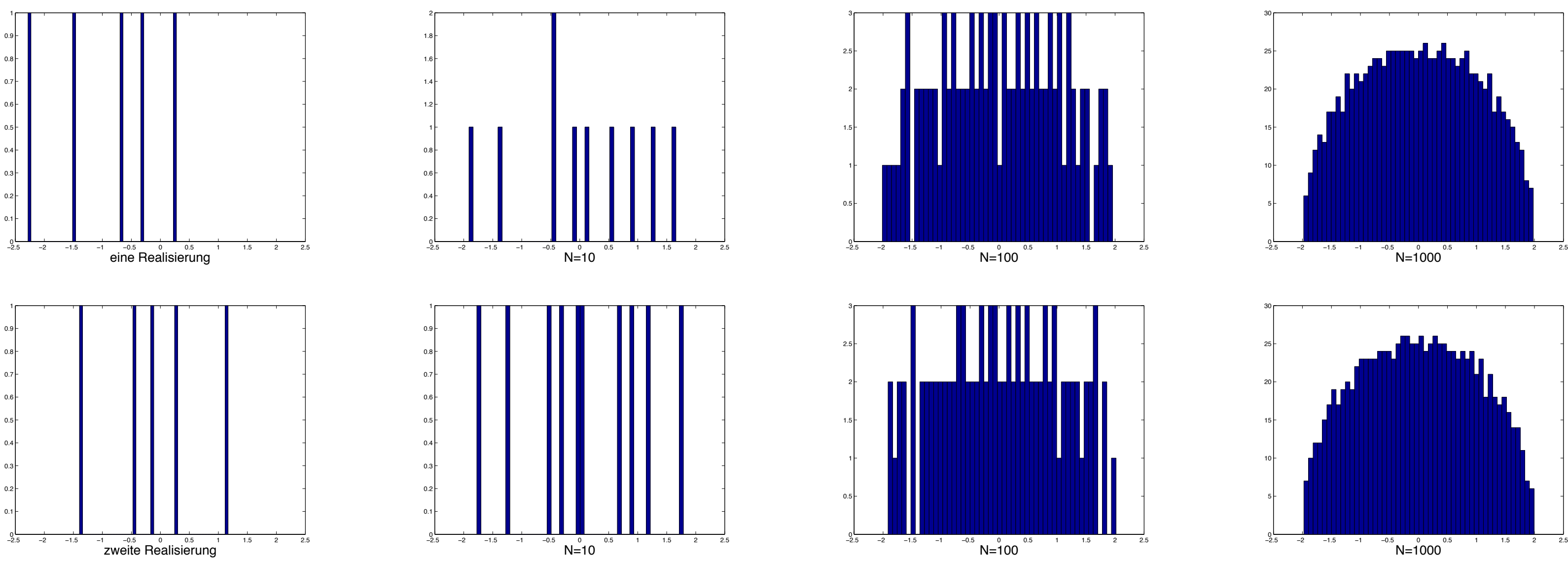
N=10



N=100



N=1000


eine Realisierung
N=10
N=100
N=1000
zweite Realisierung
N=10
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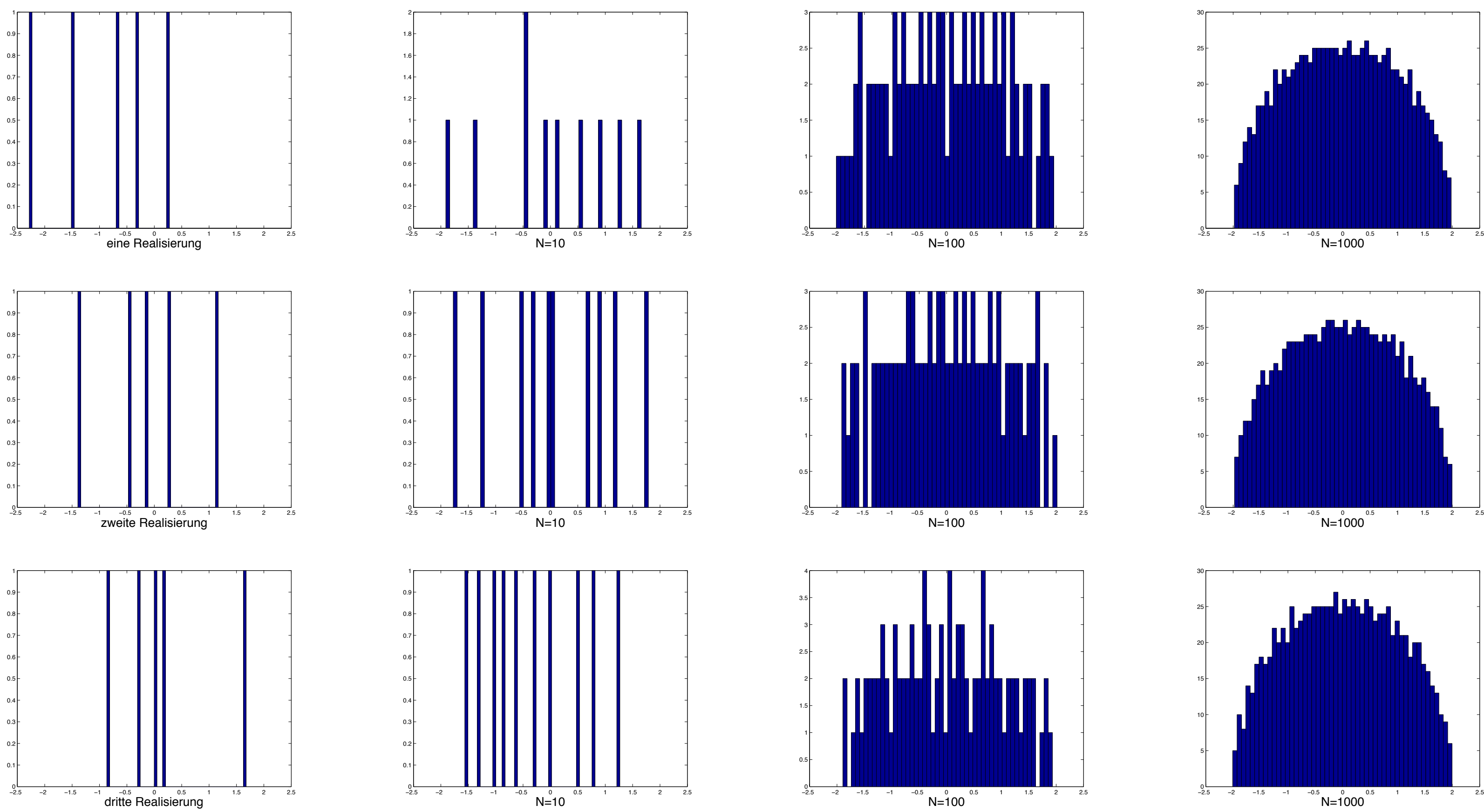

eine Realisierung
N=10
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dritte Realisierung
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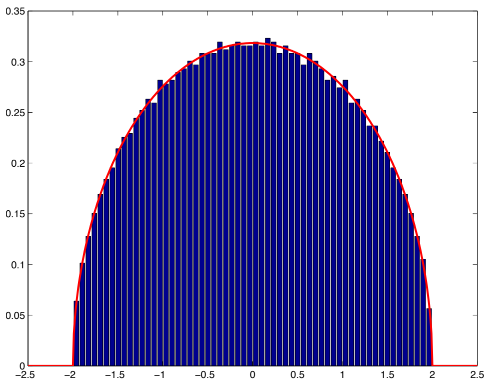
N=1000

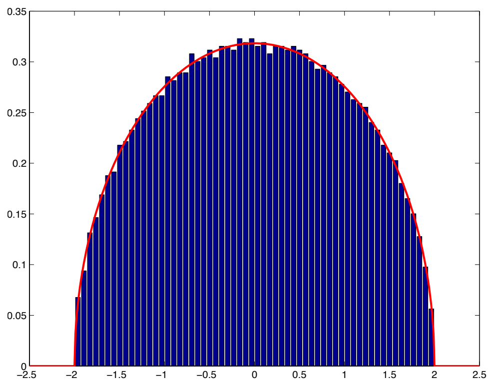
Consider selfadjoint **Gaussian $N \times N$ random matrix**.

We have almost sure convergence (convergence of "typical" realization) of its eigenvalue distribution to

Wigner's semicircle.



... one realization ...

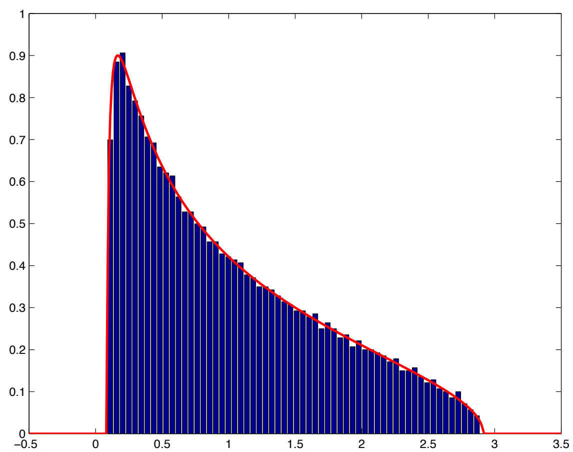


... another realization ...

$$N = 4000$$

Consider **Wishart random matrix** $A = XX^*$, where X is $N \times M$ random matrix with independent Gaussian entries.
Its eigenvalue distribution converges almost surely to

Marchenko-Pastur distribution.



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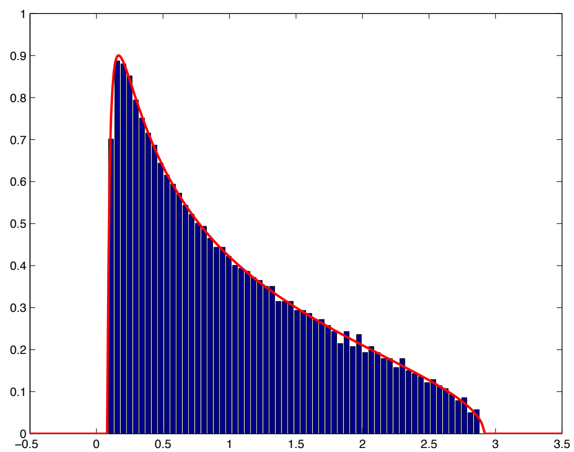

... another realization ...

$$N = 3000, M = 6000$$

We want to consider more complicated situations, built out of simple cases (like Gaussian or Wishart) by doing operations like

- taking the sum of two matrices
- taking the product of two matrices
- taking corners of matrices

Note: If several $N \times N$ random matrices A and B are involved then the eigenvalue distribution of non-trivial functions $f(A, B)$ (like $A + B$ or AB) will of course depend on the relation between the eigenspaces of A and of B.

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However: we might expect that we have almost sure convergence to a deterministic result

- if $\boldsymbol{N \to \infty}$ and

- if the eigenspaces are almost surely in a **"typical" or "generic" position**.

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- if $N \to \infty$ and

- if the eigenspaces are almost surely in a **"typical" or "generic" position**.

This is the realm of **free probability theory**.

Consider $N \times N$ random matrices A and B such that

- A has an asymptotic eigenvalue distribution for $N \to \infty$
  B has an asymptotic eigenvalue distribution for $N \to \infty$

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Then, almost surely, eigenspaces of A and of B are in generic position.

In such a generic case we expect that the asymptotic eigenvalue distribution of functions of A and B should almost surely depend in a deterministic way on the asymptotic eigenvalue distribution of A and of B the asymptotic eigenvalue distribution.

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Basic examples for such functions:

- the sum

$$A + B$$

- the product

$$AB$$

- corners of the unitarily invariant matrix B

Example: sum of independent Gaussian and Wishart ($M = 2N$) random matrices, for $N = 3000$

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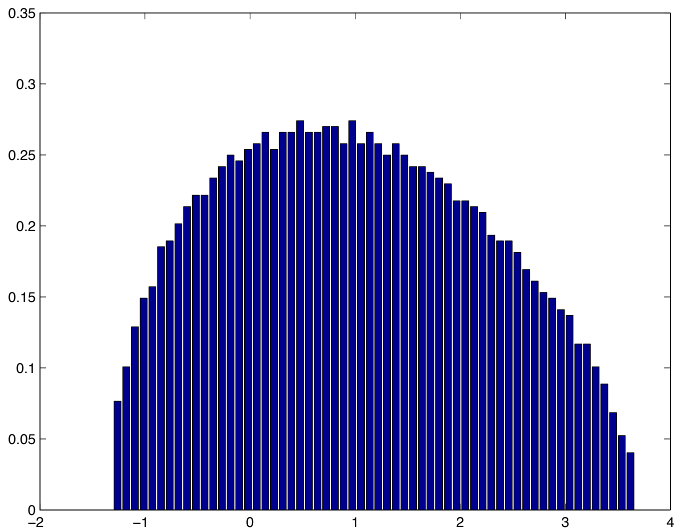

... one realization ...

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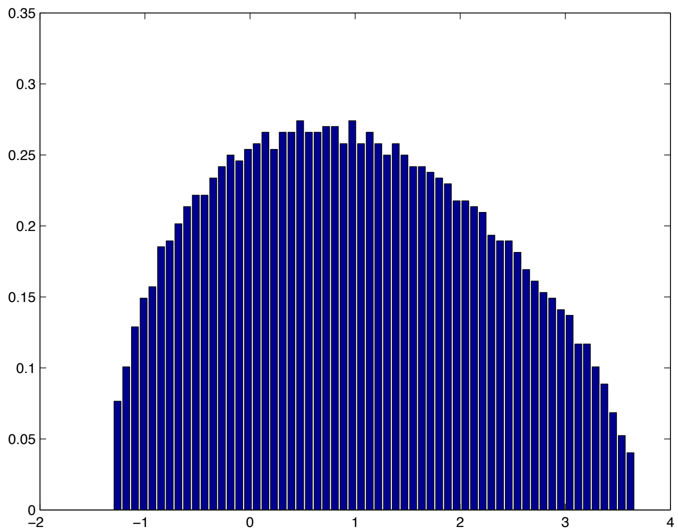

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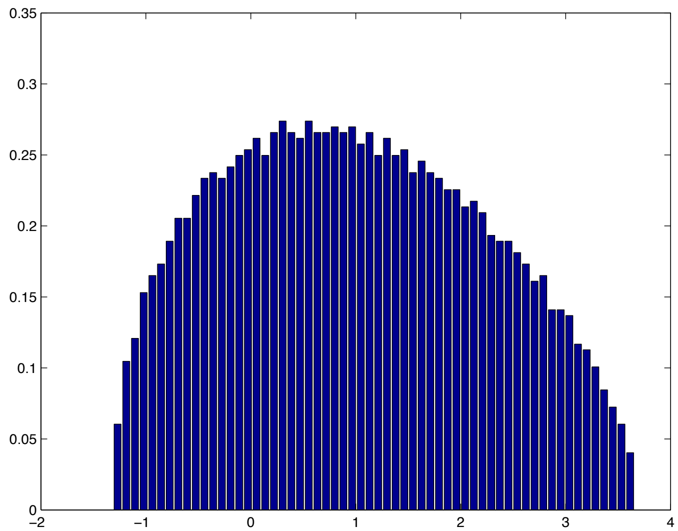

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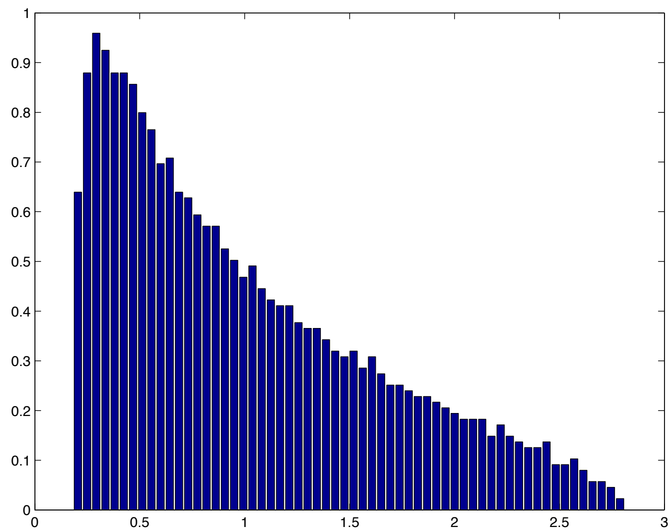

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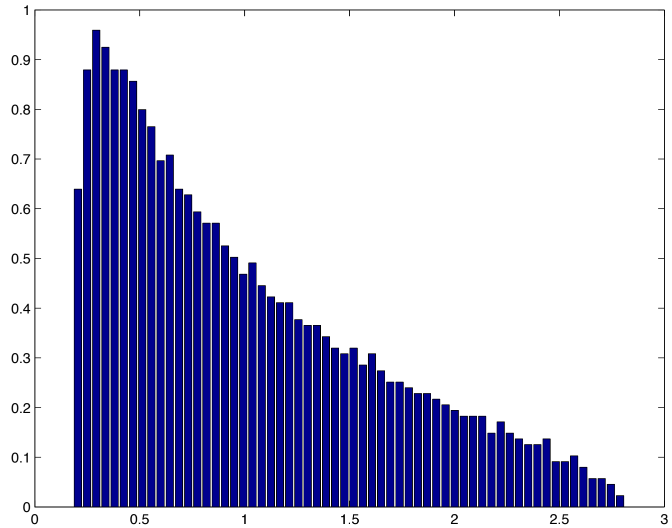

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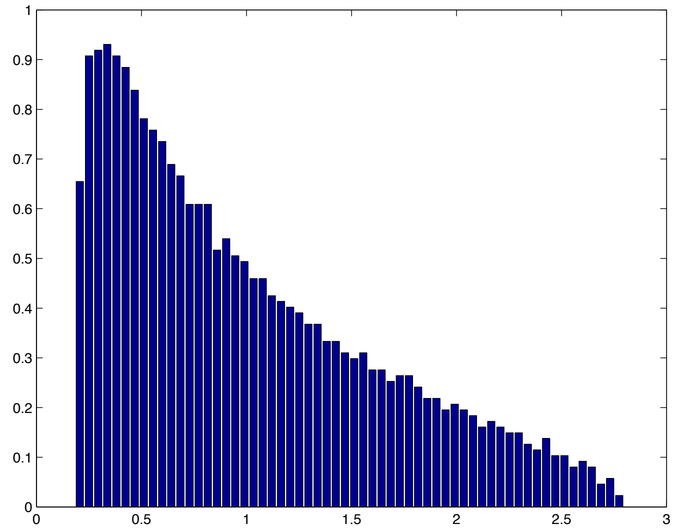

... another realization ...

Example: upper left corner of size $N/2 \times N/2$ of a randomly rotated $N \times N$ projection matrix,
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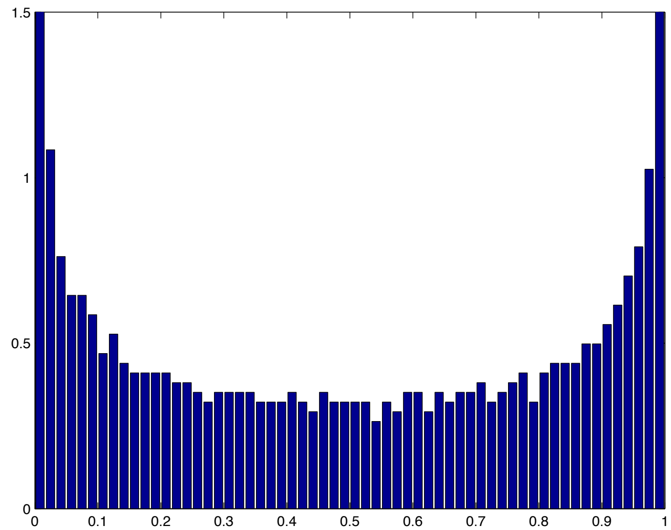

... one realization ...

$N = 2048$

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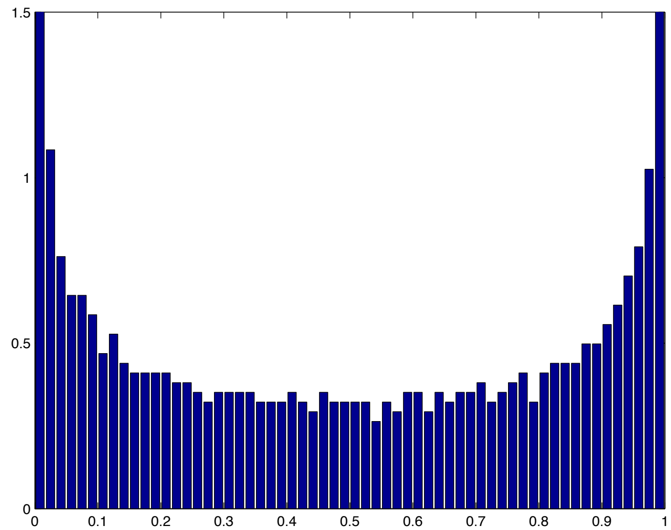

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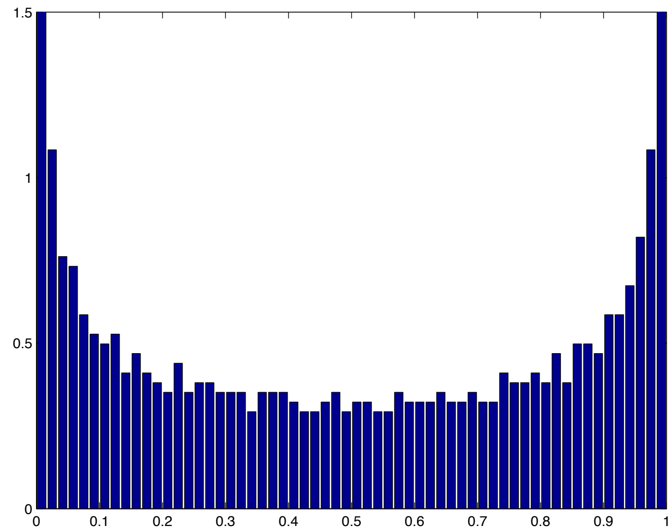

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Problems:

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- Is there an **algorithm for actually calculating** the corresponding asymptotic eigenvalue distributions?

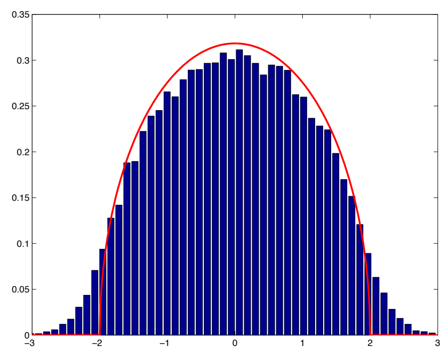
Instead of eigenvalue distribution of typical realization we will now look at eigenvalue distribution averaged over ensemble.

This has the advantages:

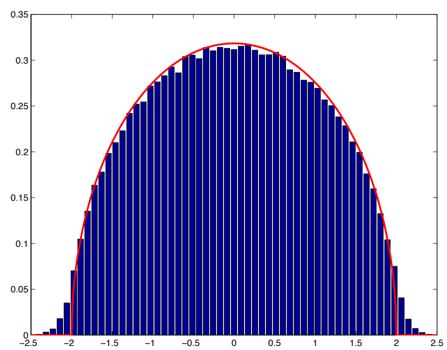
- convergence to asymptotic eigenvalue distribution happens much faster; very good agreement with asymptotic limit for moderate N

- theoretically easier to deal with averaged situation than with almost sure one (note however, this is just for convenience; the following can also be justified for typical realizations)

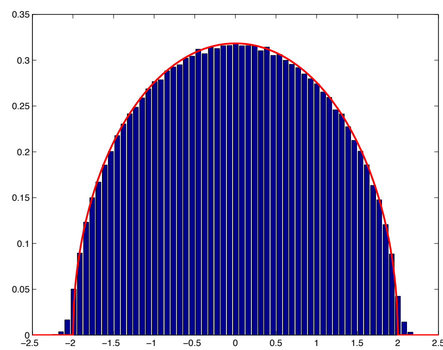
Example: Convergence of averaged eigenvalue distribution of $N \times N$ Gaussian random matrix to **semicircle**



N=5



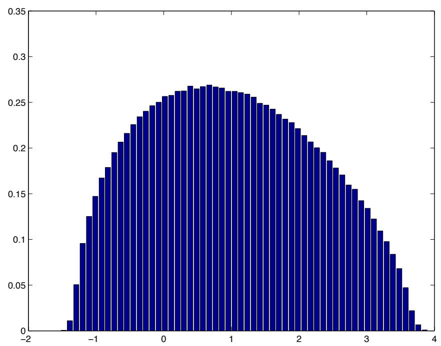
N=20



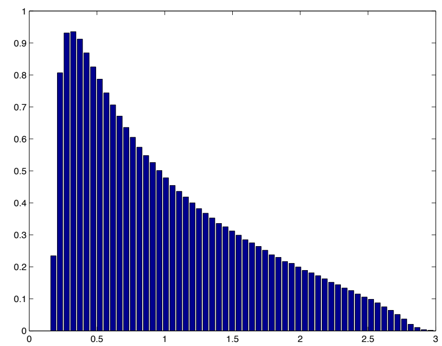
N=50

trials=10000

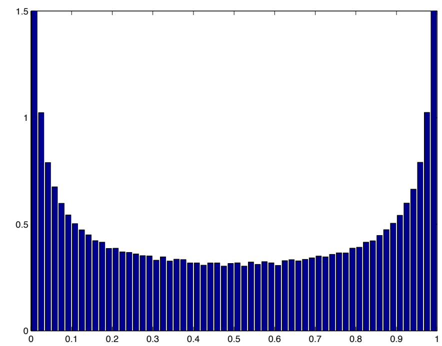
Examples: averaged sums, products, corners for moderate N



averaged Wigner + Wishart; N=100



averaged Wishart x Wishart; N=100



averaged upper corner; N=64

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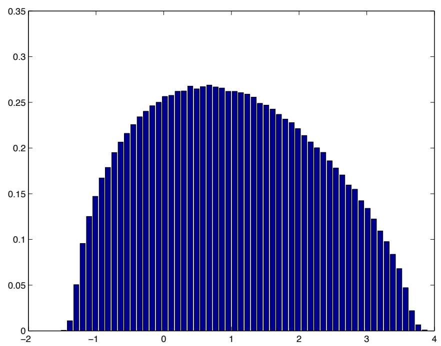

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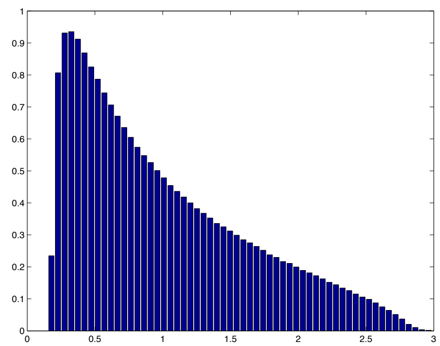
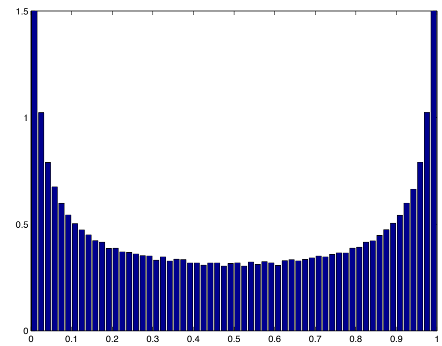


averaged Wishart x Wishart; N=100



averaged upper corner; N=64

What is the asymptotic eigenvalue distribution in these cases?

How does one analyze asymptotic eigenvalue distributions?

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- analytical

- combinatorial

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- combinatorial: moment method
  try to calculate moments of the limit distribution
  advantage: can, in principle, deal directly with several matrices A, B; by looking on **mixed moments**

Moment Method

eigenvalue distribution of matrix A	$\widehat{=}$	knowledge of traces of powers, $\mathrm{tr}(A^k)$
$\frac{1}{N}\left(\lambda_1^k + \cdots + \lambda_N^k\right)$	$=$	$\mathrm{tr}(A^k)$
averaged eigenvalue distribution of random matrix A	$\widehat{=}$	knowledge of expectations of traces of powers, $E[\mathrm{tr}(A^k)]$

Moment Method

Consider random matrices A and B in generic position.

We want to understand $f(A, B)$ in a uniform way for many f!

We have to understand for all $k \in \mathbb{N}$ the moments

$$E\left[\text{tr}\Big(f(A, B)^k\Big)\right].$$

Moment Method

Consider random matrices A and B in generic position.

We want to understand $A+B$, AB, $AB-BA$, etc.

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$$E\left[\text{tr}\Big((A+B)^k\Big)\right], \quad E\left[\text{tr}\Big((AB)^k\Big)\right], \quad E\left[\text{tr}\Big((AB-BA)^k\Big)\right], \quad \text{etc.}$$

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Thus we need to understand as basic objects

mixed moments $\qquad E\left[\text{tr}\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)\right]$

Use following notation:

$$\varphi(A) := \lim_{N\to\infty} E[\mathrm{tr}(A)].$$

Question: If A and B are in generic position, can we understand

$$\varphi\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)$$

in terms of

$$\Big(\varphi(A^k)\Big)_{k\in\mathbb{N}} \qquad \text{and} \qquad \Big(\varphi(B^k)\Big)_{k\in\mathbb{N}}$$

Example: independent Gaussian random matrices

Consider two independent Gaussian random matrices A and B

Then, in the limit $N \to \infty$, the moments

$$\varphi(A^{n_1} B^{m_1} A^{n_2} B^{m_2} \cdots)$$

are given by

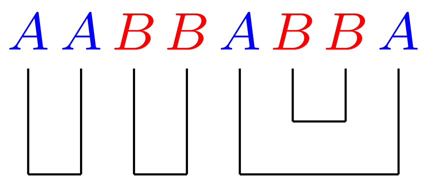
$$\#\Big\{\text{non-crossing/planar pairings of pattern}$$

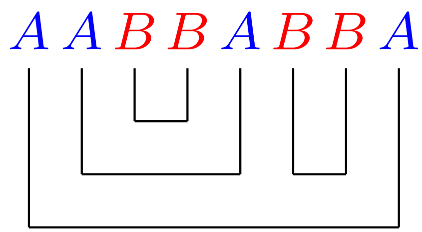
$$\underbrace{A \cdot A \cdots A}_{n_1\text{-times}} \cdot \underbrace{B \cdot B \cdots B}_{m_1\text{-times}} \cdot \underbrace{A \cdot A \cdots A}_{n_2\text{-times}} \cdot \underbrace{B \cdot B \cdots B}_{m_2\text{-times}} \cdots,$$

$$\text{which do not pair } A \text{ with } B\Big\}$$

Example: $\varphi(AABBABBA) = 2$

because there are two such non-crossing pairings:

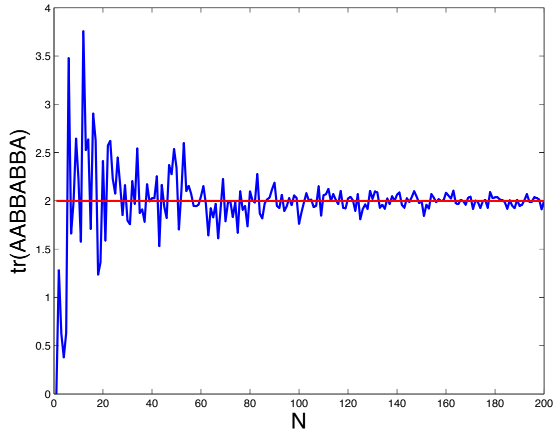


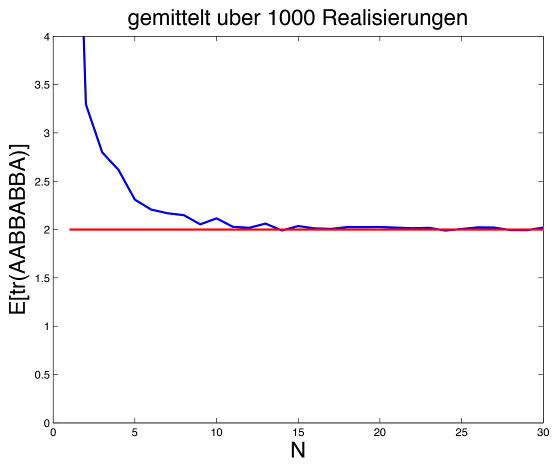


Example: $\varphi(AABBABBA) = 2$

one realization

averaged over 1000 realizations





$$\varphi\left(A^{n_1} B^{m_1} A^{n_2} B^{m_2} \cdots\right)$$

$$= \#\left\{\text{non-crossing pairings which do not pair } A \text{ with } B\right\}$$

$$\varphi\left(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\right)$$

$$= \#\Big\{\text{non-crossing pairings which do not pair } A \text{ with } B\Big\}$$

implies

$$\varphi\Big(\big(A^{n_1}-\varphi(A^{n_1})\cdot 1\big)\cdot\big(B^{m_1}-\varphi(B^{m_1})\cdot 1\big)\cdot\big(A^{n_2}-\varphi(A^{n_2})\cdot 1\big)\cdots\Big)$$

$$= \#\Big\{\text{non-crossing pairings which do not pair } A \text{ with } B,$$
$$\text{and for which each blue group and each red group is}$$
$$\text{connected with some other group}\Big\}$$

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$$= \#\Big\{\text{non-crossing pairings which do not pair } A \text{ with } B\Big\}$$

implies

$$\varphi\Big(\big(A^{n_1} - \varphi(A^{n_1})\cdot 1\big)\cdot\big(B^{m_1} - \varphi(B^{m_1})\cdot 1\big)\cdot\big(A^{n_2} - \varphi(A^{n_2})\cdot 1\big)\cdots\Big)$$

$$= 0$$

Actual equation for the calculation of the mixed moments

$$\varphi_1 \left(A^{n_1} B^{m_1} A^{n_2} B^{m_2} \cdots \right)$$

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is different for different random matrix ensembles.

However, the relation between the mixed moments,

$$\varphi\Big(\big(A^{n_1}-\varphi(A^{n_1})\cdot 1\big)\cdot\big(B^{m_1}-\varphi(B^{m_1})\cdot 1\big)\cdots\Big)=0$$

remains the same for matrix ensembles in generic position and constitutes the **definition of freeness**.

Definition [Voiculescu 1985]: A and B are **free** (with respect to φ) if we have for all $n_1, m_1, n_2, \dots \geq 1$ that

$$\varphi\Big(\big(A^{n_1} - \varphi(A^{n_1}) \cdot 1\big) \cdot \big(B^{m_1} - \varphi(B^{m_1}) \cdot 1\big) \cdot \big(A^{n_2} - \varphi(A^{n_2}) \cdot 1\big) \cdots\Big) = 0$$

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$$\varphi\Big(\text{alternating product in centered words in } A \text{ and in } B\Big) = 0$$

Theorem [Voiculescu 1991]: Consider $N \times N$ random matrices A and B such that

- A has an asymptotic eigenvalue distribution for $N \to \infty$
  B has an asymptotic eigenvalue distribution for $N \to \infty$

- A and B are independent
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- B is a unitarily invariant ensemble
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Then, for $N \to \infty$, A and B are free.

Definition of Freeness

Let $(\mathcal{A}, \varphi)$ be **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$)

Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever

- $a_i \in \mathcal{A}_{j(i)}, \qquad j(i) \in I \quad \forall i, \qquad j(1) \neq j(2) \neq \cdots \neq j(n)$

- $\varphi(a_i) = 0 \quad \forall i$

Random variables $x_1, \dots, x_n \in \mathcal{A}$ are free, if their generated unital subalgebras $\mathcal{A}_i :=$ algebra$(1, x_i)$ are so.

What is Freeness?

Freeness between A and B is an infinite set of equations relating various moments in A and B:

$$\varphi\Big(p_1(A)q_1(B)p_2(A)q_2(B)\cdots\Big) = 0$$

Basic observation: freeness between A and B is actually a **rule for calculating mixed moments** in A and B from the moments of A and the moments of B:

$$\varphi\Big(A^{n_1}B^{m_1}A^{n_2}B^{m_2}\cdots\Big) = \text{polynomial}\Big(\varphi(A^i), \varphi(B^j)\Big)$$

Example:

$$\varphi\Big(\big(A^n - \varphi(A^n)1\big)\big(B^m - \varphi(B^m)1\big)\Big) = 0,$$

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thus

$$\varphi(A^n B^m) - \varphi(A^n \cdot 1)\varphi(B^m) - \varphi(A^n)\varphi(1 \cdot B^m) + \varphi(A^n)\varphi(B^m)\varphi(1 \cdot 1) = 0,$$

Example:

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thus

$$\varphi(A^n B^m) - \varphi(A^n \cdot 1)\varphi(B^m) - \varphi(A^n)\varphi(1 \cdot B^m) + \varphi(A^n)\varphi(B^m)\varphi(1 \cdot 1) = 0,$$

and hence

$$\boldsymbol{\varphi(A^n B^m) = \varphi(A^n) \cdot \varphi(B^m)}$$

Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables.

Thus freeness is also called **free independence**

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Note: free independence is a different rule from classical independence; free independence occurs typically for **non-commuting random variables**, like operators on Hilbert spaces or (random) matrices

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Example:

$$\varphi\Big(\big(A-\varphi(A)1\big)\cdot\big(B-\varphi(B)1\big)\cdot\big(A-\varphi(A)1\big)\cdot\big(B-\varphi(B)1\big)\Big)=0,$$

which results in

$$\begin{aligned}\varphi(ABAB) = \varphi(AA)\cdot\varphi(B)\cdot\varphi(B) + \varphi(A)\cdot\varphi(A)\cdot\varphi(BB) \\ - \varphi(A)\cdot\varphi(B)\cdot\varphi(A)\cdot\varphi(B)\end{aligned}$$

Motivation for the combinatorics of freeness: the free (and classical) CLT

Consider $a_1, a_2, \dots \in (\mathcal{A}, \varphi)$ which are

- identically distributed
- centered and normalized: $\varphi(a_i) = 0$ and $\varphi(a_i^2) = 1$
- either classically independent or freely independent

What can we say about

$$S_n := \frac{a_1 + \cdots + a_n}{\sqrt{n}} \quad \xrightarrow{n \to \infty} \quad ???$$

We say that S_n converges (in distribution) to s if

$$\varphi(S_n^m) = \varphi(s^m) \qquad \forall m \in \mathbb{N}$$

We have

$$\varphi(S_n^m) = \frac{1}{n^{m/2}} \varphi\Big[(a_1 + \cdots a_n)^m\Big]$$

$$= \frac{1}{n^{m/2}} \sum_{i(1),\ldots,i(m)=1}^{n} \varphi[a_{i(1)} \cdots a_{i(m)}]$$

Note:

$$\varphi[a_{i(1)} \cdots a_{i(m)}] = \varphi[a_{j(1)} \cdots a_{j(m)}]$$

whenever

$$\ker i = \ker j$$

For example, $i = (1,3,1,5,3)$ and $j = (3,4,3,6,4)$:

$$\varphi[a_1 a_3 a_1 a_5 a_3] = \varphi[a_3 a_4 a_3 a_6 a_4]$$

because independence/freeness allows to express

$$\varphi[a_1 a_3 a_1 a_5 a_3] = \text{polynomial}\Big(\varphi(a_1), \varphi(a_1^2), \varphi(a_3), \varphi(a_3^2), \varphi(a_5)\Big)$$
$$\varphi[a_3 a_4 a_3 a_6 a_4] = \text{polynomial}\Big(\varphi(a_3), \varphi(a_3^2), \varphi(a_4), \varphi(a_4^2), \varphi(a_6)\Big)$$

and

$$\varphi(a_1) = \varphi(a_3), \qquad \varphi(a_1^2) = \varphi(a_3^2)$$

$$\varphi(a_3) = \varphi(a_4), \qquad \varphi(a_3^2) = \varphi(a_4^2), \qquad \varphi(a_5) = \varphi(a_6)$$

We put

$\kappa_\pi := \varphi[a_1 a_3 a_1 a_5 a_3]$ where $\pi := \ker i = \ker j = \{\{1,3\},\{2,5\},\{4\}\}$

$\pi \in \mathcal{P}(5)$ is a partition of $\{1,2,3,4,5\}$.

Thus

$$\varphi(S_n^m) = \frac{1}{n^{m/2}} \sum_{i(1),\dots,i(m)=1}^{n} \varphi[a_{i(1)} \cdots a_{i(m)}]$$

$$= \frac{1}{n^{m/2}} \sum_{\pi \in \mathcal{P}(m)} \kappa_\pi \cdot \#\{i : \ker i = \pi\}$$

Note:

$$\#\{i : \ker i = \pi\} \quad = \quad n(n-1)\cdots(n-\#\pi-1) \sim n^{\#\pi}$$

So

$$\varphi(S_n^m) \sim \sum_{\pi \in \mathcal{P}(m)} \kappa_\pi \cdot n^{\#\pi - m/2}$$

No singletons in the limit

Consider $\pi \in \mathcal{P}(m)$ with singleton:

$$\pi = \{\ldots, \{k\}, \ldots\},$$

thus

$$\kappa_\pi = \varphi(a_{i(1)} \cdots a_{i(k)} \cdots a_{i(m)})$$

$$= \varphi(a_{i(1)} \cdots a_{i(k-1)} a_{i(k+1)} \cdots a_{i(m)}) \cdot \underbrace{\varphi(a_{i(k)})}_{=0}$$

Thus: $\kappa_\pi = 0$ if π has singleton; i.e.,

$$\kappa_\pi \neq 0 \quad \Longrightarrow \quad \pi = \{V_1, \ldots, V_r\} \text{ with } \#V_j \geq 2 \, \forall j$$

$$\Longrightarrow \quad r = \#\pi \leq \frac{m}{2}$$

So in

$$\varphi(S_n^m) \sim \sum_{\pi \in \mathcal{P}(m)} \kappa_\pi \cdot n^{\#\pi - m/2}$$

only those π survive for $n \to \infty$ with

- π has no singleton, i.e., no block of size 1

- π has exactly $m/2$ blocks

Such π are exactly those, where each block has size 2, i.e.,

$$\pi \in \mathcal{P}_2(m) := \{\pi \in \mathcal{P}(m) \mid \pi \text{ is pairing}\}$$

Thus we have:

$$\lim_{n\to\infty} \varphi(S_n^m) = \sum_{\pi\in\mathcal{P}_2(m)} \kappa_\pi$$

In particular: odd moments are zero (because no pairings of odd number of elements), thus limit distribution is symmetric

Question: What are the even moments?

This depends on the κ_π's.

The actual value of those is now different for the classical and the free case!

Classical CLT: assume a_i are independent

If the a_i commute and are independent, then

$$\kappa_\pi = \varphi(a_{i(1)} \cdots a_{i(2k)}) = 1 \qquad \forall\, \pi \in \mathcal{P}_2(2k)$$

Thus

$$\lim_{n\to\infty} \varphi(S_n^m) = \#\mathcal{P}_2(m) = \begin{cases} 0, & m \text{ odd} \\ (m-1)(m-3)\cdots 5\cdot 3\cdot 1, & m \text{ even} \end{cases}$$

Those limit moments are the moments of a Gaussian distribution of variance 1.

Free CLT: assume a_i are free

If the a_i are free, then, for $\pi \in \mathcal{P}_2(2k)$,

$$\kappa_\pi = \begin{cases} 0, & \pi \text{ is crossing} \\ 1, & \pi \text{ is non-crossing} \end{cases}$$

E.g.,

$$\begin{aligned} \kappa_{\{1,6\},\{2,5\},\{3,4\}} &= \varphi(a_1 a_2 a_3 a_3 a_2 a_1) \\ &= \varphi(a_3 a_3) \cdot \varphi(a_1 a_2 a_2 a_1) \\ &= \varphi(a_3 a_3) \cdot \varphi(a_2 a_2) \cdot \varphi(a_1 a_1) \\ &= 1 \end{aligned}$$

but

$$\begin{aligned} \kappa_{\{1,5\},\{2,3\},\{4,6\}} &= \varphi(a_1 a_2 a_2 a_3 a_1 a_3) \\ &= \varphi(a_2 a_2) \cdot \underbrace{\varphi(a_1 a_3 a_1 a_3)}_{0} \end{aligned}$$

Free CLT: assume a_i are free

Put

$$NC_2(m) := \{\pi \in \mathcal{P}_2(m) \mid \pi \text{ is non-crossing}\}$$

Thus

$$\lim_{n\to\infty} \varphi(S_n^m) = \#NC_2(m) = \begin{cases} 0, & m \text{ odd} \\ c_k = \frac{1}{k+1}\binom{2k}{k}, & m = 2k \text{ even} \end{cases}$$

Those limit moments are the moments of a semicircular distribution of variance 1,

$$\lim_{n\to\infty} \varphi(S_n^m) = \frac{1}{2\pi}\int_{-2}^{2} t^m \sqrt{4-t^2}\,dt$$

How to recognize the Catalan numbers c_k

Put

$$c_k := \#NC_2(2k).$$

We have

$$c_k = \sum_{\pi \in NC(2k)} 1 = \sum_{i=1}^{k} \sum_{\pi = \{1,2i\} \cup \pi_1 \cup \pi_2} 1 = \sum_{i=1}^{k} c_{i-1} c_{k-i}$$

This recursion, together with $c_0 = 1, c_1 = 1$, determines the sequence of Catalan numbers:

$$\{c_k\} = 1, 1, 2, 5, 14, 42, 132, 429, \ldots.$$

Understanding the freeness rule: the idea of cumulants

- write moments in terms of other quantities, which we call **free cumulants**

- freeness is much easier to describe on the level of free cumulants: **vanishing of mixed cumulants**

- relation between moments and cumulants is given by summing over **non-crossing or planar partitions**

Non-crossing partitions

A **partition** of $\{1,\ldots,n\}$ is a decomposition $\pi = \{V_1,\ldots,V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq y), \qquad \bigcup_i V_i = \{1,\ldots,n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.

π is **non-crossing** if we do not have

$$p_1 < q_1 < p_2 < q_2$$

such that p_1, p_2 are in same block, q_1, q_2 are in same block, but those two blocks are different.

$$\mathbf{NC(n)} := \{\text{non-crossing partitions of } \{1,\ldots,n\}\}$$

$NC(n)$ is actually a lattice with refinement order.

Moments and cumulants

For unital linear functional

$$\varphi : \mathcal{A} \to \mathbb{C}$$

we define **cumulant functionals** κ_n (for all $n \geq 1$)

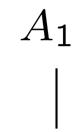
$$\kappa_n : \mathcal{A}^n \to \mathbb{C}$$

as multi-linear functionals by moment-cumulant relation

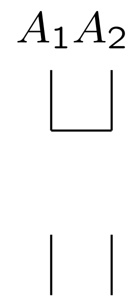
$$\varphi(A_1 \cdots A_n) = \sum_{\pi \in NC(n)} \kappa_\pi[A_1, \ldots, A_n]$$

Note: classical cumulants are defined by a similar formula, where only $NC(n)$ is replaced by $\mathcal{P}(n)$

$$\varphi(A_1) = \kappa_1(A_1)$$



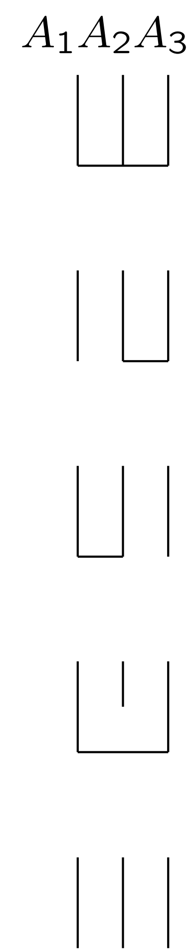
$$\varphi(A_1 A_2) = \kappa_2(A_1, A_2) + \kappa_1(A_1)\kappa_1(A_2)$$



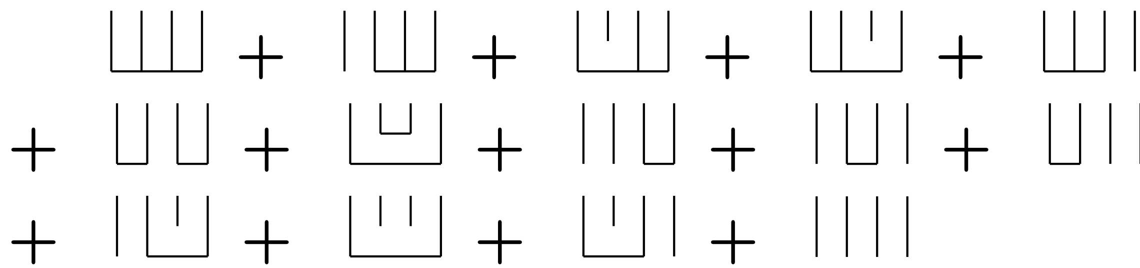
thus

$$\kappa_2(A_1, A_2) = \varphi(A_1 A_2) - \varphi(A_1)\varphi(A_2)$$

$$
\begin{aligned}
\varphi(A_1A_2A_3) = {} & \kappa_3(A_1, A_2, A_3) \\
& + \kappa_1(A_1)\kappa_2(A_2, A_3) \\
& + \kappa_2(A_1, A_2)\kappa_1(A_3) \\
& + \kappa_2(A_1, A_3)\kappa_1(A_2) \\
& + \kappa_1(A_1)\kappa_1(A_2)\kappa_1(A_3)
\end{aligned}
$$



$$\varphi(A_1A_2A_3A_4) =$$



$$\begin{aligned}
= \; & \kappa_4(A_1,A_2,A_3,A_4) + \kappa_1(A_1)\kappa_3(A_2,A_3,A_4) \\
& + \kappa_1(A_2)\kappa_3(A_1,A_3,A_4) + \kappa_1(A_3)\kappa_3(A_1,A_2,A_4) \\
& + \kappa_3(A_1,A_2,A_3)\kappa_1(A_4) + \kappa_2(A_1,A_2)\kappa_2(A_3,A_4) \\
& + \kappa_2(A_1,A_4)\kappa_2(A_2,A_3) + \kappa_1(A_1)\kappa_1(A_2)\kappa_2(A_3,A_4) \\
& + \kappa_1(A_1)\kappa_2(A_2,A_3)\kappa_1(A_4) + \kappa_2(A_1,A_2)\kappa_1(A_3)\kappa_1(A_4) \\
& + \kappa_1(A_1)\kappa_2(A_2,A_4)\kappa_1(A_3) + \kappa_2(A_1,A_4)\kappa_1(A_2)\kappa_1(A_3) \\
& + \kappa_2(A_1,A_3)\kappa_1(A_2)\kappa_1(A_4) + \kappa_1(A_1)\kappa_1(A_2)\kappa_1(A_3)\kappa_1(A_4)
\end{aligned}$$

Freeness $\hat{=}$ vanishing of mixed cumulants

Theorem [Speicher 1994]: The fact that A and B are free is equivalent to the fact that

$$\kappa_n(C_1, \dots, C_n) = 0$$

whenever

- $n \geq 2$
- $C_i \in \{A, B\}$ for all i
- there are i, j such that $C_i = A$, $C_j = B$

Freeness $\hat{=}$ vanishing of mixed cumulants

free product $\hat{=}$ direct sum of cumulants

$\varphi(\boldsymbol{A}^{\boldsymbol{n}})$ given by sum over **blue** planar diagrams

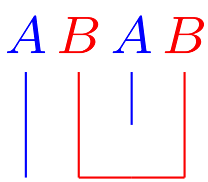
$\varphi(\boldsymbol{B}^{\boldsymbol{m}})$ given by sum over **red** planar diagrams

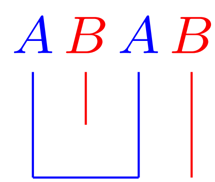
then: for A and B free, $\varphi(\boldsymbol{A}^{\boldsymbol{n_1}}\boldsymbol{B}^{\boldsymbol{m_1}}\boldsymbol{A}^{\boldsymbol{n_2}}\cdots)$ is given by sum over planar diagrams with monochromatic (blue or red) blocks

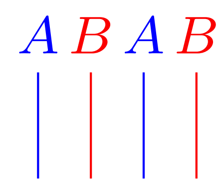
Vanishing of Mixed Cumulants

$$\varphi(ABAB) =$$

$$\kappa_1(A)\kappa_1(A)\kappa_2(B,B)+\kappa_2(A,A)\kappa_1(B)\kappa_1(B)+\kappa_1(A)\kappa_1(B)\kappa_1(A)\kappa_1(B)$$





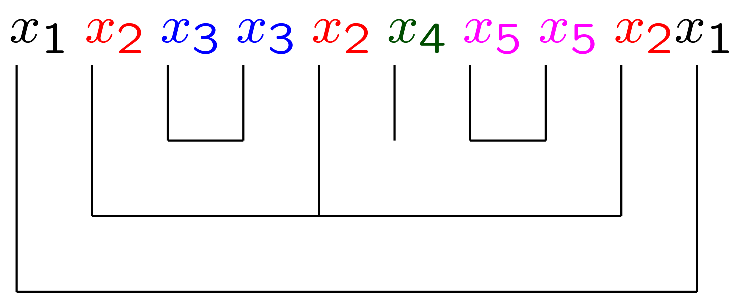


Factorization of Non-Crossing Moments

The iteration of the rule

$$\varphi(A_1 B A_2) = \varphi(A_1 A_2)\varphi(B) \qquad \text{if } \{A_1, A_2\} \text{ and } B \text{ free}$$

leads to the factorization of all "non-crossing" moments in free variables



$$\varphi(x_1 x_2 x_3 x_3 x_2 x_4 x_5 x_5 x_2 x_1)$$

$$= \varphi(x_1 x_1) \cdot \varphi(x_2 x_2 x_2) \cdot \varphi(x_3 x_3) \cdot \varphi(x_4) \cdot \varphi(x_5 x_5)$$

Sum of Free Variables

Consider A, B free.

Then, by freeness, the moments of $A+B$ are uniquely determined by the moments of A and the moments of B.

Notation: We say the distribution of $A+B$ is the

free convolution

of the distribution of A and the distribution of B,

$$\mu_{A+B} = \mu_A \boxplus \mu_B.$$

Sum of Free Variables

In principle, freeness determines this, but the concrete nature of this rule on the level of moments is not apriori clear.

Example:

$$\varphi\Big((A+B)^1\Big) = \varphi(A) + \varphi(B)$$

$$\varphi\Big((A+B)^2\Big) = \varphi(A^2) + 2\varphi(A)\varphi(B) + \varphi(B^2)$$

$$\varphi\Big((A+B)^3\Big) = \varphi(A^3) + 3\varphi(A^2)\varphi(B) + 3\varphi(A)\varphi(B^2) + \varphi(B^3)$$

$$\begin{aligned}\varphi\Big((A+B)^4\Big) = {}& \varphi(A^4) + 4\varphi(A^3)\varphi(B) + 4\varphi(A^2)\varphi(B^2) \\ & + 2\Big(\varphi(A^2)\varphi(B)\varphi(B) + \varphi(A)\varphi(A)\varphi(B^2) \\ & \quad - \varphi(A)\varphi(B)\varphi(A)\varphi(B)\Big) + 4\varphi(A)\varphi(B^3) + \varphi(B^4)\end{aligned}$$

Sum of Free Variables

Corresponding rule on level of free cumulants is easy: If A and B are free then

$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)$$
$$+\kappa_n(\dots, A, B, \dots) + \cdots$$

Sum of Free Variables

Corresponding rule on level of free cumulants is easy: If A and B are free then

$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)$$

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$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)$$

i.e., we have **additivity of cumulants for free variables**

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

Sum of Free Variables

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But: how well do we understand the relation between moments and free cumulants???

Sum of Free Variables

Corresponding rule on level of free cumulants is easy: If A and B are free then

$$\kappa_n(A+B, A+B, \dots, A+B) = \kappa_n(A, A, \dots, A) + \kappa_n(B, B, \dots, B)$$

i.e., we have additivity of cumulants for free variables

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B$$

Combinatorial relation between moments and cumulants can be rewritten easily as a relation between corresponding formal power series.

Relation between moments and free cumulants

We have

$$m_n := \varphi(A^n) \qquad \text{moments}$$

and

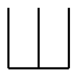
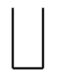
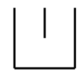
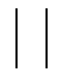
$$\kappa_n := \kappa_n(A, A, \dots, A) \qquad \text{free cumulants}$$

Combinatorially, the relation between them is given by

$$m_n = \varphi(A^n) = \sum_{\pi \in NC(n)} \kappa_\pi$$

Example:

$$m_1 = \kappa_1, \quad m_2 = \kappa_2 + \kappa_1^2, \quad m_3 = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3$$

$m_3 = \kappa$  $+ \kappa$  $+ \kappa$  $+ \kappa$  $+ \kappa$  $= \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3$

Theorem [Speicher 1994]: Consider formal power series

$$M(z) = 1 + \sum_{k=1}^{\infty} m_n z^n, \qquad C(z) = 1 + \sum_{k=1}^{\infty} \kappa_n z^n$$

Then the relation

$$m_n = \sum_{\pi \in NC(n)} \kappa_\pi$$

between the coefficients is equivalent to the relation

$$M(z) = C[zM(z)]$$

Proof

First we get the following recursive relation between cumulants and moments

$$m_n = \sum_{\pi \in NC(n)} \kappa_\pi$$

$$= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} \sum_{\pi_1 \in NC(i_1)} \cdots \sum_{\pi_s \in NC(i_s)} \kappa_s \kappa_{\pi_1} \cdots \kappa_{\pi_s}$$

$$= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} \kappa_s m_{i_1} \cdots m_{i_s}$$

$$m_n = \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} \kappa_s m_{i_1} \cdots m_{i_s}$$

Plugging this into the formal power series $M(z)$ gives

$$M(z) = 1 + \sum_n m_n z^n$$

$$= 1 + \sum_n \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\cdots+i_s+s=n}} k_s z^s m_{i_1} z^{i_1} \cdots m_{i_s} z^{i_s}$$

$$= 1 + \sum_{s=1}^{\infty} \kappa_s z^s \Big(M(z) \Big)^s = C[zM(z)]$$

Remark on classical cumulants

Classical cumulants c_k are combinatorially defined by

$$m_n = \sum_{\pi \in \mathcal{P}(n)} c_\pi$$

In terms of generating power series

$$\tilde{M}(z) = 1 + \sum_{n=1}^{\infty} \frac{m_n}{n!} z^n, \qquad \tilde{C}(z) = \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n$$

this is equivalent to

$$\tilde{C}(z) = \log \tilde{M}(z)$$

From moment series to Cauchy transform

Instead of $M(z)$ we consider **Cauchy transform**

$$G(z) := \varphi(\frac{1}{z-A}) = \int \frac{1}{z-t} d\mu_A(t) = \sum \frac{\varphi(A^n)}{z^{n+1}} = \frac{1}{z} M(1/z)$$

and instead of $C(z)$ we consider **R-transform**

$$R(z) := \sum_{n \geq 0} \kappa_{n+1} z^n = \frac{C(z) - 1}{z}$$

Then $M(z) = C[zM(z)]$ becomes

$$R[G(z)] + \frac{1}{G(z)} = z \qquad \text{or} \qquad G[R(z) + 1/z] = z$$

Consider one random variable $A \in \mathcal{A}$ and define their **Cauchy transform G** and their **$\mathcal{R}$-transform $\mathcal{R}$** by

$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi(A^n)}{z^{n+1}}, \qquad \mathcal{R}(z) = \sum_{n=1}^{\infty} \kappa_n(A, \dots, A) z^{n-1}$$

Theorem [Voiculescu 1986, Speicher 1994]: Then we have

- $\frac{1}{G(z)} + \mathcal{R}(G(z)) = z$

- $\mathcal{R}^{A+B}(z) = \mathcal{R}^A(z) + \mathcal{R}^B(z)$ if A and B are free

What is advantage of $G(z)$ over $M(z)$?

For any probability measure μ is its Cauchy transform

$$G(z) := \int \frac{1}{z-t} d\mu(t)$$

an analytic function $G : \mathbb{C}^+ \to \mathbb{C}^-$ and we can recover μ from G by **Stieltjes inversion formula**

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \Im G(t + i\varepsilon) dt$$

Example: semicircular distribution μ_s

μ_s has moments given by the Catalan numbers or, equivalently, has cumulants

$$\kappa_n = \begin{cases} 0, & n \neq 2 \\ 1, & n = 2 \end{cases}$$

(because $m_n = \sum_{\pi \in NC_2(n)} \kappa_\pi$ says that $\kappa_\pi = 0$ for $\pi \in NC(n)$ which is not a pairing), thus

$$R(z) = \sum_{n \geq 0} \kappa_{n+1} z^n = \kappa_2 \cdot z = z$$

and hence

$$z = R[G(z)] + \frac{1}{G(z)} = G(z) + \frac{1}{G(z)}$$

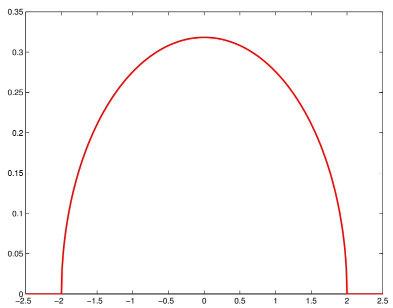
or $$G(z)^2 + 1 = zG(z)$$

$$G(z)^2 + 1 = zG(z) \qquad \text{thus} \qquad G(z) = \frac{z \pm \sqrt{z^4 - 4}}{2}$$

We have "-", because $G(z) \sim 1/z$ for $z \to \infty$; then

$$d\mu_s(t) = -\frac{1}{\pi}\Im\Big(\frac{t - \sqrt{t^2 - 4}}{2}\Big)dt$$

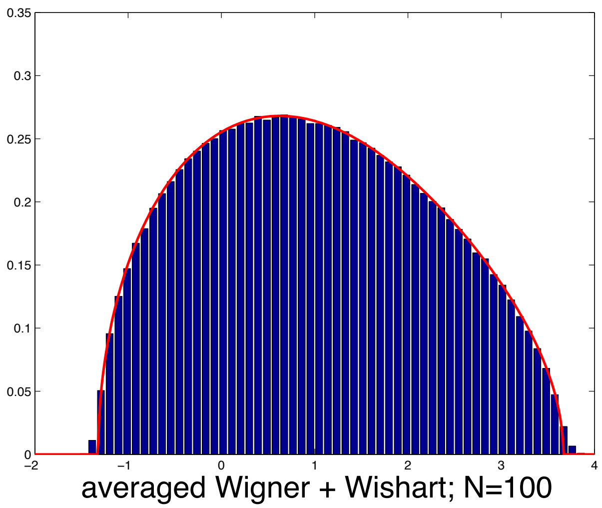
$$= \begin{cases} \frac{1}{2\pi}\sqrt{4 - t^2}dt, & \text{if } t \in [-2, 2] \\ 0, & \text{otherwise} \end{cases}$$

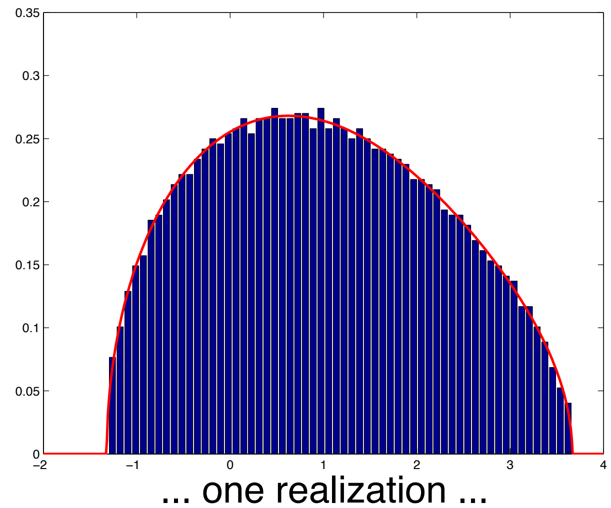


The additivity of the R-transform, together with the relation between Cauchy transform and $\mathcal{R}$-transform and the the Stieltjes inversion formula, gives an effective algorithm for calculating free convolutions, i.e., the asymptotic eigenvalue distribution **of sums of random matrices in generic position**:

$$
\begin{array}{ccccccccccc}
A & \rightsquigarrow & G^A & \rightsquigarrow & R^A & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & R^A & +R^B = R^{A+B} & \rightsquigarrow & G^{A+B} & \rightsquigarrow & A+B \\
 & & & & \uparrow & & & & & & \\
B & \rightsquigarrow & G^B & \rightsquigarrow & R^B & & & & & &
\end{array}
$$

Example: Wigner + Wishart ($M = 2N$)





trials=4000

N=3000

What is the free Binomial $\left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}\right)^{\boxplus 2}$

$$\mu := \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}, \qquad \nu := \mu \boxplus \mu$$

Then $$G_\mu(z) = \int \frac{1}{z-t} d\mu(t) = \frac{1}{2}\Big(\frac{1}{z+1} + \frac{1}{z-1}\Big) = \frac{z}{z^2-1}$$

and so $$z = G_\mu[R_\mu(z) + 1/z] = \frac{R_\mu(z) + 1/z}{(R_\mu(z) + 1/z)^2 - 1}$$

thus $$R_\mu(z) = \frac{\sqrt{1+4z^2} - 1}{2z}$$

and so $$R_\nu(z) = 2R_\mu(z) = \frac{\sqrt{1+4z^2} - 1}{z}$$

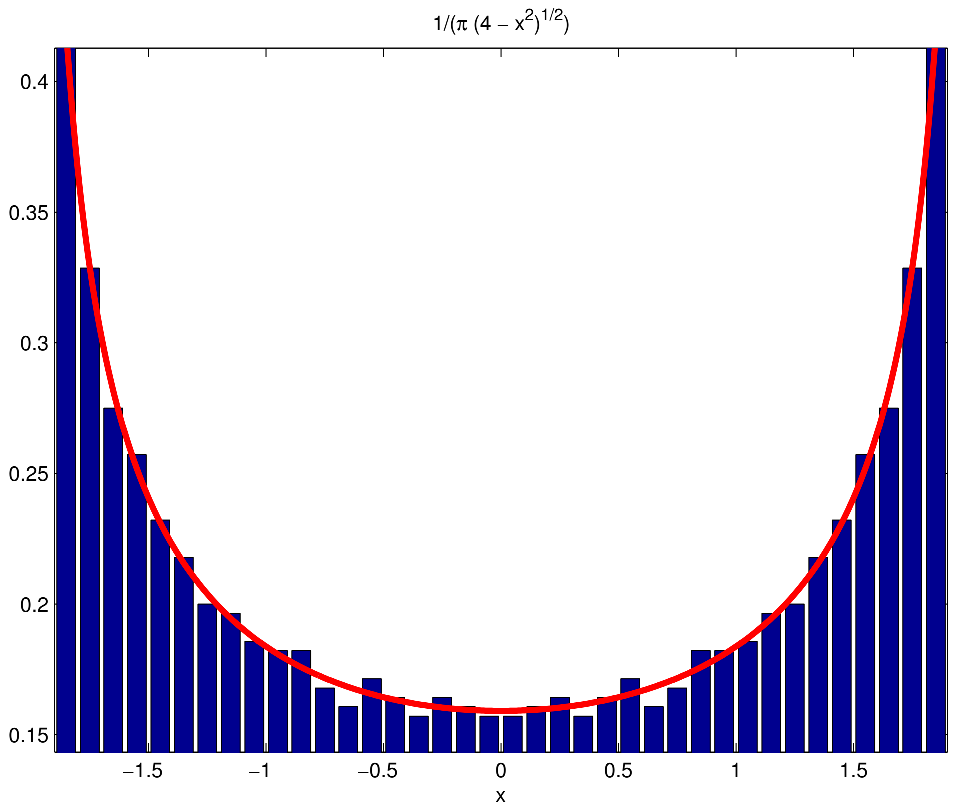
$$R_\nu(z) = \frac{\sqrt{1+4z^2}-1}{z} \qquad \text{gives} \qquad G_\nu(z) = \frac{1}{\sqrt{z^2-4}}$$

and thus

$$d\nu(t) = -\frac{1}{\pi}\Im\frac{1}{\sqrt{t^2-4}}dt = \begin{cases} \frac{1}{\pi\sqrt{4-t^2}}, & |t| \le 2 \\ 0, & \text{otherwise} \end{cases}$$

So

$$\left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}\right)^{\boxplus 2} = \nu = \text{arcsine-distribution}$$



2800 eigenvalues of $A + UBU^*$, where A and B are diagonal matrices with 1400 eigenvalues +1 and 1400 eigenvalues -1, and U is a randomly chosen unitary matrix

Lessons to learn

Free convolution of discrete distributions is in general non discrete

Since it is true that

$$\delta_x \boxplus \delta_y = \delta_{x+y}$$

we see, in particular, that $\boxplus$ is not linear, i.e, for $\alpha + \beta = 1$

$$(\alpha\mu_1 + \beta\mu_2) \boxplus \nu \neq \alpha(\mu_1 \boxplus \nu) + \beta(\mu_2 \boxplus \nu)$$

Non-commutativity matters: the sum of two commuting projections is a quite easy object, the sum of two non-commuting projections is much harder to grasp

The R-transform as an Analytic Object

- The R-transform can be established as an analytic function via power series expansions around the point infinity in the complex plane.

- The R-transform can, in contrast to the Cauchy transform, in general not be defined on all of the upper complex half-plane, but only in some truncated cones (which depend on the considered variable).

- The equation $\frac{1}{G(z)} + R[G(z)] = z$ does in general not allow explicit solutions and there is no good numerical algorithm for dealing with this.

An Alternative to the R-transform: Subordination

Let x and y be free. Put $w := R_{x+y}(z) + 1/z$, then

$$G_{x+y}(w) = z = G_x[R^x(z){+}1/z] = G_x[w{-}R_y(z)] = G_x[w{-}R_y[G_{x+y}(w)]]$$

Thus with

$$\omega(z) := z - R_y[G_{x+y}(z)]]$$

we have the subordination

$$G_{x+y}(z) = G_x\Big(\omega(z)\Big)$$

The subordination function ω has good analytic properties!

The Subordination Function

Let x and y be free. Put

$$F(z) := \frac{1}{G(z)}$$

Then there exists an analytic $\omega : \mathbb{C}^+ \to \mathbb{C}^+$ such that

$$F_{x+y}(z) = F_x\Big(\omega(z)\Big) \qquad \text{and} \qquad G_{x+y}(z) = G_x\Big(\omega(z)\Big)$$

The subordination function $\omega(z)$ is given as the unique fixed point in the upper half-plane of the map

$$f_z(w) = F_y(F_x(w) - w + z) - (F_x(w) - w)$$

Product of Free Variables

Consider A, B free.

Then, by freeness, the moments of AB are uniquely determined by the moments of A and the moments of B.

Notation: We say the distribution of AB is the

free multiplicative convolution

of the distribution of A and the distribution of B,

$$\mu_{AB} = \mu_A \boxtimes \mu_B.$$

Caveat: AB not selfadjoint

Note: $\boxtimes$ is in general not operation on probability measures on $\mathbb{R}$. Even if both A and B are selfadjoint, AB is only selfadjoint if A and B commute (which they don't, if they are free)

But: if B is positive, then we can consider $B^{1/2}AB^{1/2}$ instead of AB. Since A and B are free it follows that both have the same moments; e.g.,

$$\varphi(B^{1/2}AB^{1/2}) = \varphi(B^{1/2}B^{1/2})\varphi(A) = \varphi(B)\varphi(A) = \varphi(AB)$$

So the "right" definition is

$$\mu_A \boxtimes \mu_B = \mu_{B^{1/2}AB^{1/2}}.$$

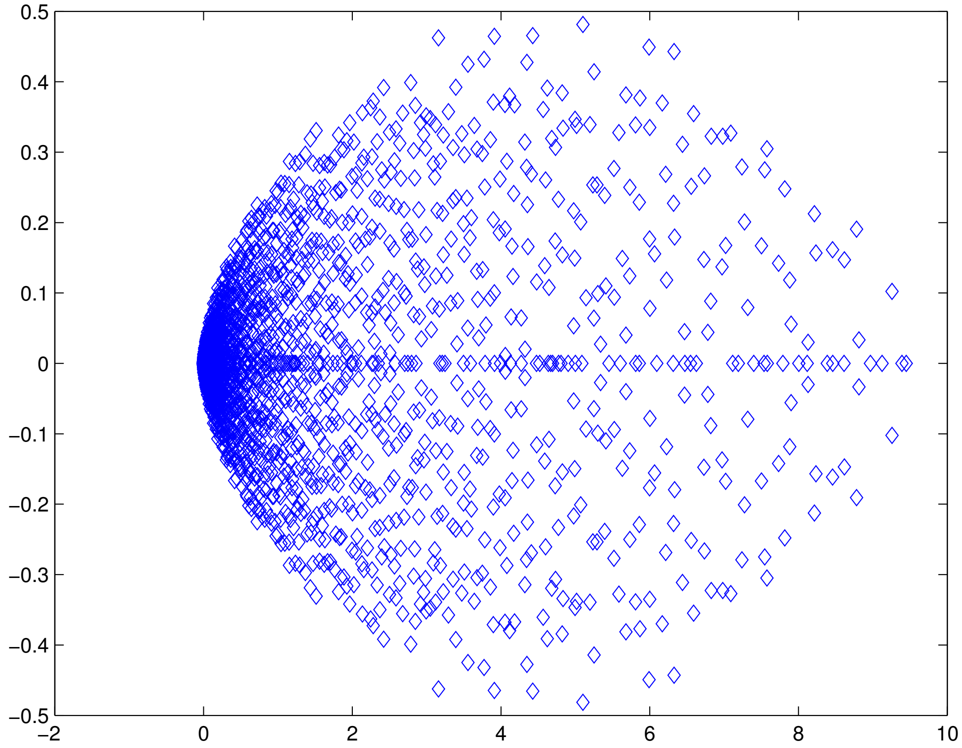
If we also restrict A to be positive, then this gives a binary operation on probability measures on $\mathbb{R}^+$

Meaning of $\boxtimes$ for random matrices

If A and B are symmetric matrices with positive eigenvalues, then the eigenvalues of $AB = (AB^{1/2})B^{1/2}$ are the same as the eigenvalues of $B^{1/2}(AB^{1/2})$ and thus are necessarily also real and positive. If A and B are asymptotically free, the eigenvalues of AB are given by $\mu_A \boxtimes \mu_B$.

However: this is not true any more for three or more matrices!!!
If A, B, C are symmetric matrices with positive eigenvalues, then there is no reason that the eigenvalues of ABC are real.

If A, B, C are asymptotically free, then still the moments of ABC are the same as the moments of $C^{1/2}B^{1/2}AB^{1/2}C^{1/2}$, i.e., the same as the moments of $\mu_A \boxtimes \mu_B \boxtimes \mu_C$. But since the eigenvalues of ABC are not real, the knowledge of the moments is not enough to determine them.



3000 complex eigenvalues of the product of three independent 3000×3000 Wishart matrices

Back to the general theory of ⊠

In principle, the moments of AB are, for A and B free, determined by the moments of A and the moments of B; but again the concrete nature of this rule on the level of moments is not clear...

$$\varphi\Big((AB)^1\Big) = \varphi(A)\varphi(B)$$

$$\varphi\Big((AB)^2\Big) = \varphi(A^2)\varphi(B)^2 + \varphi(A)^2\varphi(B^2) - \varphi(A)^2\varphi(B)^2$$

$$\begin{aligned}\varphi\Big((AB)^3\Big) = {} & \varphi(A^3)\varphi(B)^3 + \varphi(A)^3\varphi(B^3) + 3\varphi(A)\varphi(A^2)\varphi(B)\varphi(B^2) \\ & - 3\varphi(A)\varphi(A^2)\varphi(B)^3 - 3\varphi(A)^3\varphi(B)\varphi(B^2) + 2\varphi(A)^3\varphi(B)^3\end{aligned}$$

... so let's again look on cumulants.

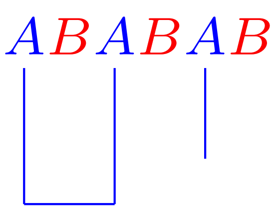
Product of Free Variables

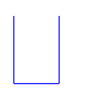
Corresponding rule on level of free cumulants is relatively easy (at least conceptually): If A and B are free then

$$\kappa_n(AB, AB, \dots, AB) = \sum_{\pi \in NC(n)} \kappa_\pi[A, A, \dots, A] \cdot \kappa_{K(\pi)}[B, B, \dots, B],$$

where $K(\pi)$ is the **Kreweras complement** of π: $K(\pi)$ is the maximal σ with

$$\pi \in NC(\text{blue}), \quad \sigma \in NC(\text{red}), \quad \pi \cup \sigma \in NC(\text{blue} \cup \text{red})$$



$K($  $) =$

Product of Free Variables

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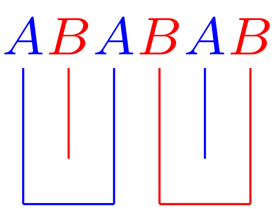
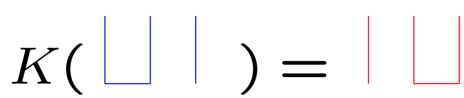
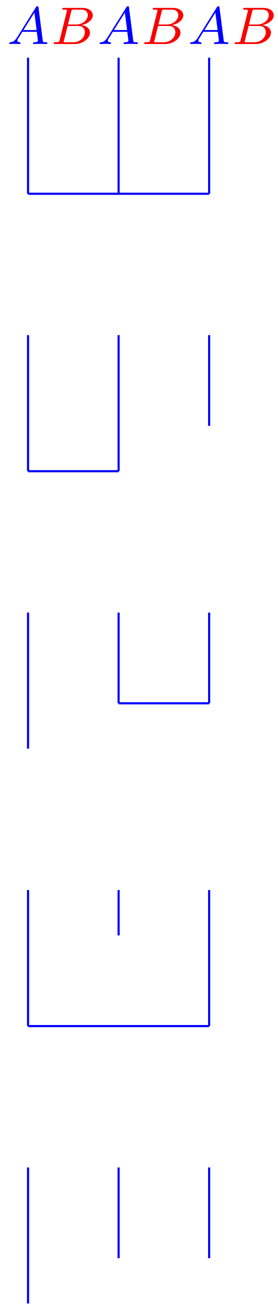
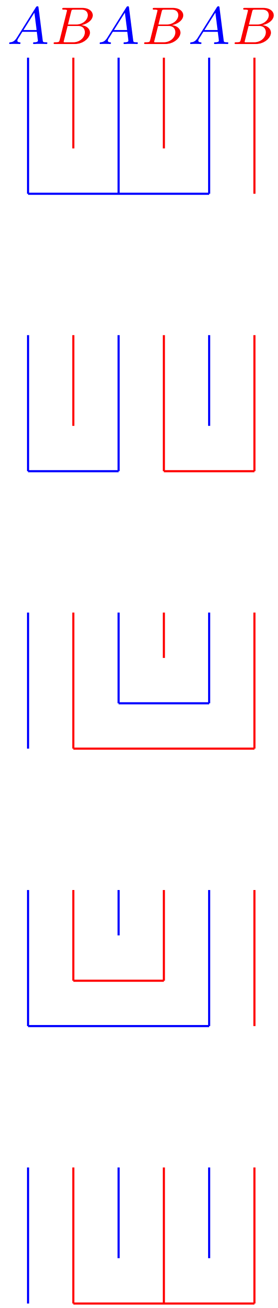
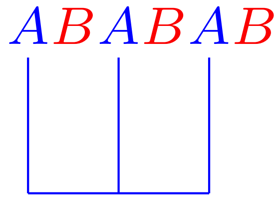


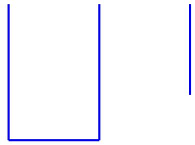



A B A B A B

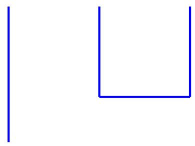

ABABAB



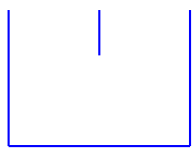
$$\kappa_3(A, A, A)$$



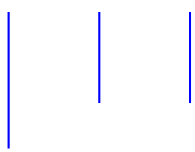
$$\kappa_2(A, A)\kappa_1(A)$$



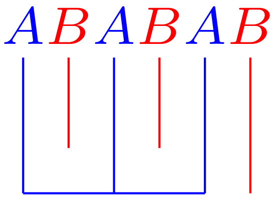
$$\kappa_2(A, A)\kappa_1(A)$$



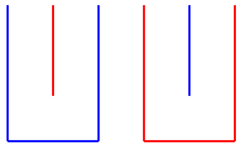
$$\kappa_2(A, A)\kappa_1(A)$$



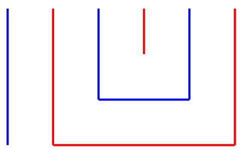
$$\kappa_1(A)\kappa_1(A)\kappa_1(A)$$



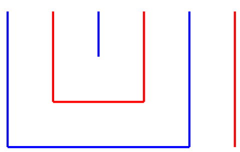
$$\kappa_3(A, A, A)\ \kappa_1(B)\kappa_1(B)\kappa_1(B)$$



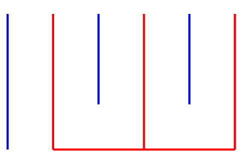
$$\kappa_2(A, A)\kappa_1(A)\ \kappa_2(B, B)\kappa_1(B)$$



$$\kappa_2(A, A)\kappa_1(A)\ \kappa_2(B, B)\kappa_1(B)$$



$$\kappa_2(A, A)\kappa_1(A)\ \kappa_2(B, B)\kappa_1(B)$$



$$\kappa_1(A)\kappa_1(A)\kappa_1(A)\ \kappa_3(B, B, B)$$

Product of Free Variables

Theorem [Voiculescu 1987; Haagerup 1997; Nica, Speicher 1997]:

Put

$$M_A(z) := \sum_{m=1}^{\infty} \varphi(A^m) z^m$$

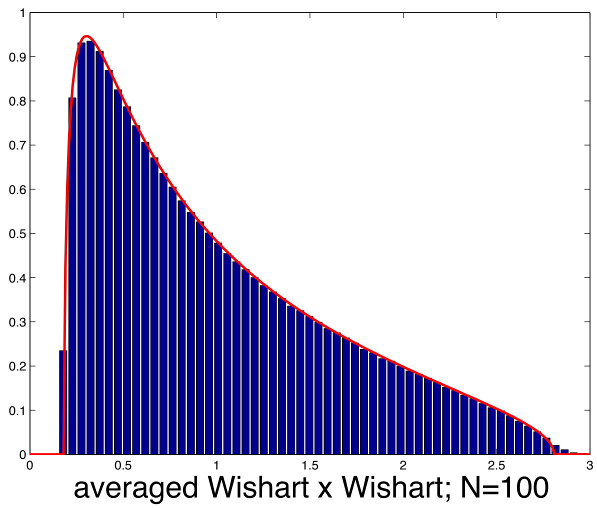
and define

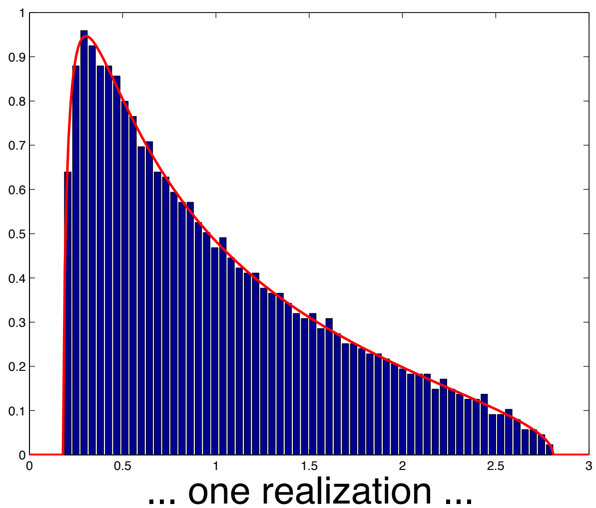
$$S_A(z) := \frac{1+z}{z} M_A^{<-1>}(z) \qquad \textbf{\textit{S}-transform of \textit{A}}$$

Then: If A and B are free, we have

$$S_{AB}(z) = S_A(z) \cdot S_B(z).$$

Example: Wishart x Wishart ($M = 5N$)





trials=10000

N=2000

Corners of Random Matrices

Theorem [Nica, Speicher 1996]: The asymptotic eigenvalue distribution of a corner B of ratio α of a unitarily invariant random matrix A is given by

$$\mu_B = \mu_{\alpha A}^{\boxplus 1/\alpha}$$

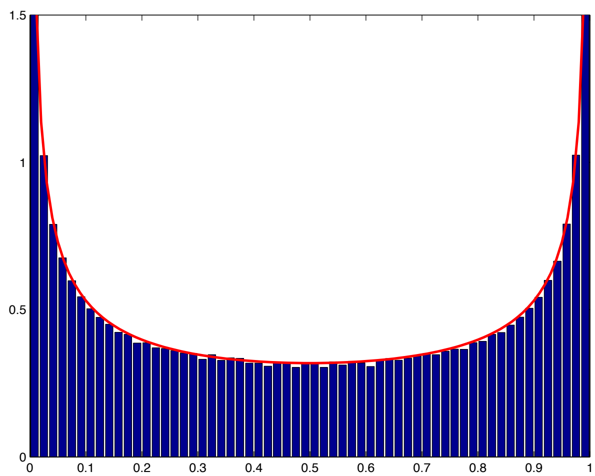
Corners of Random Matrices

Theorem [Nica, Speicher 1996]: The asymptotic eigenvalue distribution of a corner B of ratio α of a unitarily invariant random matrix A is given by

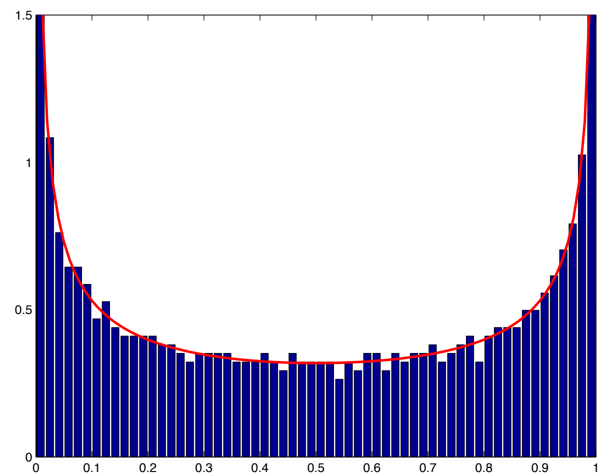
$$\mu_B = \mu_{\alpha A}^{\boxplus 1/\alpha}$$

In particular, a corner of size $\alpha = 1/2$, has up to rescaling the same distribution as $\mu_A \boxplus \mu_A$.

Upper left corner of size $N/2 \times N/2$ of a projection matrix, with $N/2$ eigenvalues 0 and $N/2$ eigenvalues 1 is, up to rescaling, the same as a free Bernoulli, i.e., the arcsine distribution



averaged upper corner; N=64



... one realization ...

trials=5000

N=2048

This actually shows

Theorem [Nica, Speicher 1996]: For any probability measure μ on $\mathbb{R}$, there exists a semigroup $(\mu^{\boxplus t})_{t\geq 1}$ of probability measures, such that $\mu^{\boxplus 1} = \mu$ and

$$\mu^{\boxplus s} \boxplus \mu^{\boxplus t} = \mu^{\boxplus (s+t)} \qquad (s, t \geq 1)$$

(Note: if this $\mu^{\boxplus t}$ exists for all $t \geq 0$, then μ is freely infinitely divisible.)

In the classical case, such a statement is not true!

Polynomials of Independent Random Matrices and Polynomials in Free Variables

We are interested in the limiting eigenvalue distribution of an

$N \times N$ random matrix for $N \to \infty$.

Typical phenomena for basic random matrix ensembles:

- almost sure convergence to a deterministic limit eigenvalue distribution

- this limit distribution can be effectively calculated

The Cauchy (or Stieltjes) Transform

For any probability measure μ on $\mathbb{R}$ we define its Cauchy transform

$$G(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t)$$

This is an analytic function $G : \mathbb{C}^+ \to \mathbb{C}^-$ and we can recover μ from G by **Stieltjes inversion formula**

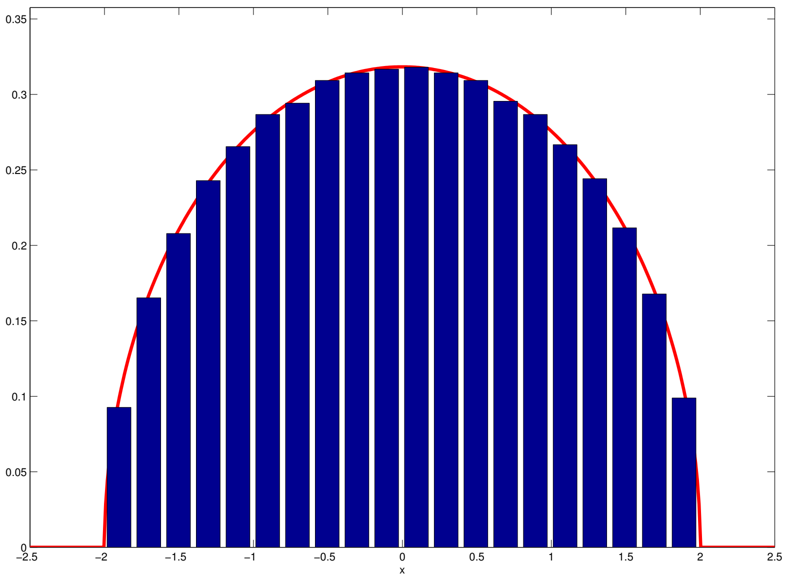
$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \Im G(t + i\varepsilon) dt$$

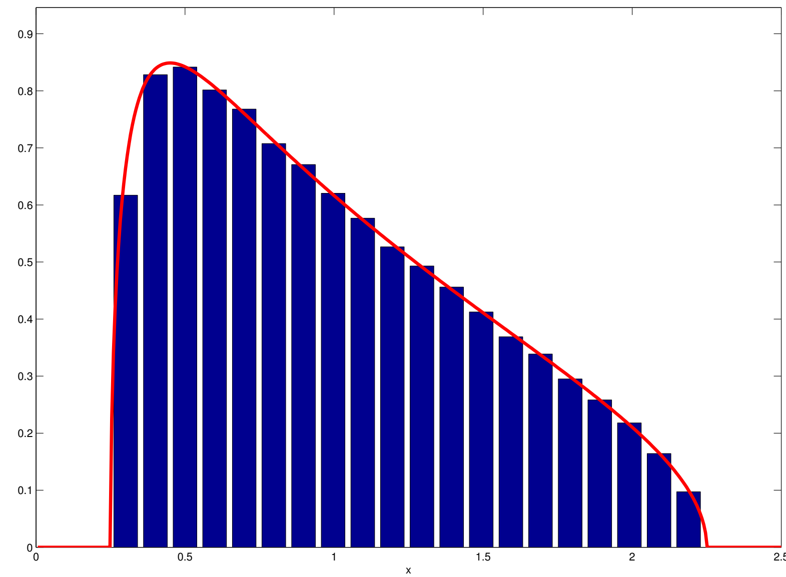
Wigner random matrix
and
Wigner's semicircle

$$G(z) = \frac{z - \sqrt{z^2 - 4}}{2}$$

Wishart random matrix
and
Marchenko-Pastur distribution

$$G(z) = \frac{z + 1 - \lambda - \sqrt{(z - (1 + \lambda))^2 - 4\lambda}}{2z}$$





We are now interested in the limiting eigenvalue distribution of

general selfadjoint polynomials $p(X_1, \dots, X_k)$

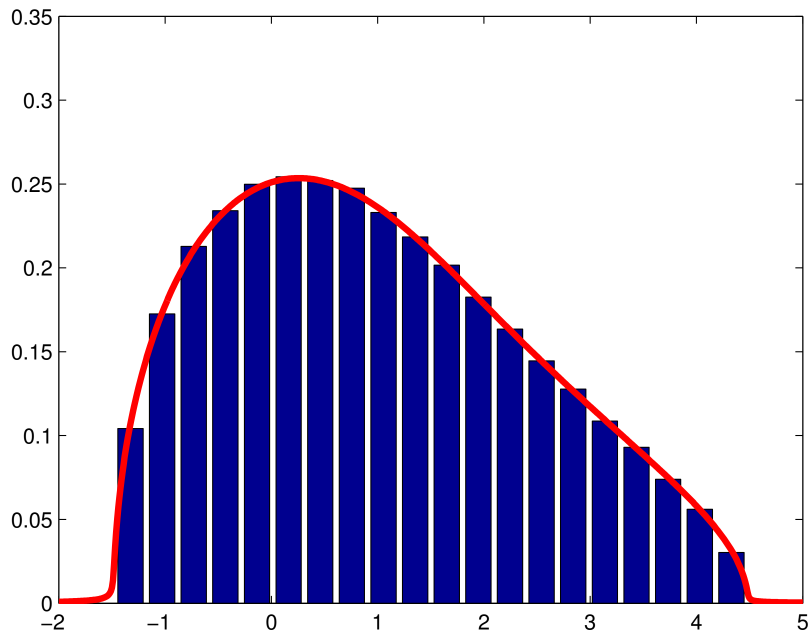
of **several** independent $N \times N$ random matrices $X_1, \dots, X_k$

Typical phenomena:

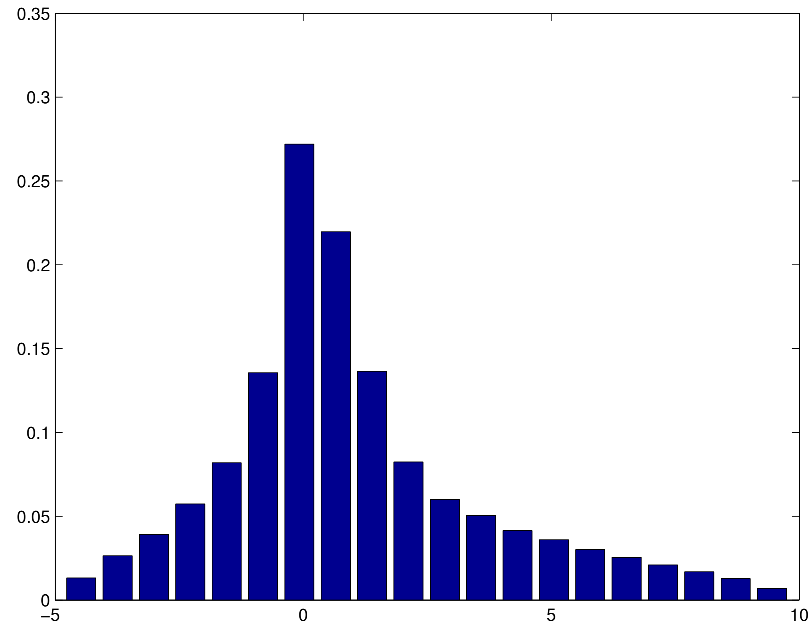
- almost sure convergence to a deterministic limit eigenvalue distribution
- this limit distribution can be effectively calculated **only in very simple situations**

for X Wigner, Y Wishart

$$p(X,Y) = X + Y$$
$$G(z) = G_{\text{Wishart}}(z - G(z))$$



$$p(X,Y) = XY + YX + X^2$$
????



Asymptotic Freeness of Random Matrices

Basic result of Voiculescu (1991):

Large classes of independent random matrices (like Wigner or Wishart matrices) become asymptoticially freely independent, with respect to $\varphi = \frac{1}{N}\text{Tr}$, if $N \to \infty$.

This means, for example: if X_N and Y_N are independent $N \times N$ Wigner and Wishart matrices, respectively, then we have almost surely:

$$\lim_{N\to\infty} \text{tr}(X_N Y_N X_N Y_N) = \lim_{N\to\infty} \text{tr}(X_N^2) \cdot \lim_{N\to\infty} \text{tr}(Y_N)^2 + \lim_{N\to\infty} \text{tr}(X_N)^2 \cdot \lim_{N\to\infty} \text{tr}(Y_N^2) - \lim_{N\to\infty} \text{tr}(X_N)^2 \cdot \lim_{N\to\infty} \text{tr}(Y_N)^2$$

Consequence: Reduction of Our Random Matrix Problem to the Problem of Polynomial in Freely Independent Variables

If the random matrices $X_1, \dots, X_k$ are asymptotically freely independent, then the distribution of a polynomial $p(X_1, \dots, X_k)$ is asymptotically given by the distribution of $p(x_1, \dots, x_k)$, where

- $x_1, \dots, x_k$ are freely independent variables, and
- the distribution of x_i is the asymptotic distribution of X_i

Can We Actually Calculate Polynomials in Freely Independent Variables?

Free probability can deal effectively with simple polynomials

- the sum of variables (Voiculescu 1986, R-transform)

$$p(x,y) = x + y$$

- the product of variables (Voiculescu 1987, S-transform)

$$p(x,y) = xy \qquad (= \sqrt{x}y\sqrt{x})$$

- the commutator of variables (Nica, Speicher 1998)

$$p(x,y) = xy - yx$$

There is no hope to calculate effectively more complicated or general polynomials in freely independent variables with usual free probability theory ...

There is no hope to calculate effectively more complicated or general polynomials in freely independent variables with usual free probability theory ...

...but there is a possible way around this: linearize the problem!!!

The Linearization Trick

The Linearization Philosophy:

In order to understand polynomials in non-commuting variables, it suffices to understand matrices of **linear** polynomials in those variables.

- Voiculescu 1987: motivation
- Haagerup, Thorbjørnsen 2005: largest eigenvalue
- Anderson 2012: the selfadjoint version
  (based on Schur complement)

- This linearization trick is also a well-known idea in many other mathematical communities, known under various names like

  Higman's trick (Higman "The units of group rings", 1940)

  * recognizable power series (automata theory, Kleene 1956, Schützenberger 1961)

  * linearization by enlargement (ring theory, Cohn 1985; Cohn and Reutenauer 1994, Malcolmson 1978)

  * descriptor realization (control theory, Kalman 1963; Helton, McCullough, Vinnikov 2006)

$\curvearrowright$ **Linearization even works for non-commutative rational functions!**

Consider a polynomial p in non-commuting variables x and y. A **linearization** of p is an $N \times N$ matrix (with $N \in \mathbb{N}$) of the form

$$\hat{p} = \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix},$$

where

- u, v, Q are matrices of the following sizes: u is $1 \times (N-1)$; v is $(N-1) \times N$; and Q is $(N-1) \times (N-1)$

- each entry of u, v, Q is a polynomial in x and y, each of degree ≤ 1

- Q is invertible and we have

$$p = -uQ^{-1}v$$

Theorem (Anderson 2012): One has

- for each p there exists a linearization $\widehat{p}$ (with an explicit algorithm for finding those)

- if p is selfadjoint, then this $\widehat{p}$ is also selfadjoint

Example of a Linearization

The selfadjoint linearization of

$$p = xy + yx + x^2 \qquad \text{is} \qquad \hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$

because we have

$$\begin{pmatrix} x & \frac{1}{2}x + y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \frac{1}{2}x + y \end{pmatrix} = -(xy + yx + x^2)$$

A linearization of $p = x_1 x_2 x_3 x_4$ is

$$\begin{pmatrix} 0 & 0 & 0 & x_1 \\ 0 & 0 & x_2 & -1 \\ 0 & x_3 & -1 & 0 \\ x_4 & -1 & 0 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 & 0 & x_1 \end{pmatrix}, Q = \begin{pmatrix} 0 & x_2 & -1 \\ x_3 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 0 \\ x_4 \end{pmatrix}$$

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Note that

$$Q^{-1} = \begin{pmatrix} 0 & x_2 & -1 \\ x_3 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -x_3 \\ -1 & -x_2 & -x_2 x_3 \end{pmatrix}$$

and thus

$$uQ^{-1}v = \begin{pmatrix} 0 & 0 & x_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -x_3 \\ -1 & -x_2 & -x_2 x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_4 \end{pmatrix} = -x_1 x_2 x_3 x_4 = -p$$

What is a Linearization Good for?

We have then

$$\hat{p} = \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix} = \begin{pmatrix} 1 & uQ^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q^{-1}v & 1 \end{pmatrix}$$

and thus (under the condition that Q is invertible):

$$p \text{ invertible} \qquad \iff \qquad \hat{p} \text{ invertible}$$

Note: $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ is always invertible with

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$$

More general, for $z \in \mathbb{C}$ put $b = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$ and then

$$b - \widehat{p} = \begin{pmatrix} z & -u \\ -v & -Q \end{pmatrix} = \begin{pmatrix} 1 & uQ^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z-p & 0 \\ 0 & -Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q^{-1}v & 1 \end{pmatrix}$$

$$z - p \text{ invertible} \qquad \Longleftrightarrow \qquad b - \widehat{p} \text{ invertible}$$

and actually

$$(b - \widehat{p})^{-1} = \left[\begin{pmatrix} 1 & uQ^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z-p & 0 \\ 0 & -Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q^{-1}v & 1 \end{pmatrix} \right]^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ -Q^{-1}v & 1 \end{pmatrix} \begin{pmatrix} (z-p)^{-1} & 0 \\ 0 & -Q^{-1} \end{pmatrix} \begin{pmatrix} 1 & -uQ^{-1} \\ 0 & 1 \end{pmatrix}$$

So

$$(b-\hat{p})^{-1} = \begin{pmatrix} 1 & 0 \\ -Q^{-1}v & 1 \end{pmatrix} \begin{pmatrix} (z-p)^{-1} & 0 \\ 0 & -Q^{-1} \end{pmatrix} \begin{pmatrix} 1 & -uQ^{-1} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (z-p)^{-1} & -(z-p)^{-1}uQ^{-1} \\ -Q^{-1}v(z-p)^{-1} & Q^{-1}v(z-p)^{-1}uQ^{-1} - Q^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (z-p)^{-1} & * \\ * & * \end{pmatrix}$$

and we can get

$$G_p(z) = \varphi((z-p)^{-1})$$

as the (1,1)-entry of the operator-valued Cauchy-transform

$$G_{\hat{p}}(b) = \mathrm{id} \otimes \varphi((b-\hat{p})^{-1}) = \begin{pmatrix} \varphi((z-p)^{-1}) & \varphi(*) \\ \varphi(*) & \varphi(*) \end{pmatrix}$$

Why is $\hat{p}$ better than p?

The selfadjoint linearization of $p = xy + yx + x^2$ is

$$\hat{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes y$$

It is a linear polynomial, but with matrix-valued coefficients.

We need to calculate its matrix-valued Cauchy transform

$$G_{\hat{p}}(b) = \mathrm{id} \otimes \varphi((b - \hat{p})^{-1})$$

with respect to

$$E = \mathrm{id} \otimes \varphi$$

Why is $\hat{p}$ better than p?

The selfadjoint linearization of $p = xy + yx + x^2$ is

$$\hat{p} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes y$$

It is a linear polynomial, but with matrix-valued coefficients.

We need to calculate its matrix-valued Cauchy transform

$$G_{\hat{p}}(b) = \mathrm{id} \otimes \varphi((b - \hat{p})^{-1})$$

Is there a matrix-valued free probability theory, with respect to the matrix-valued conditional expectation

$$E = \mathrm{id} \otimes \varphi$$

Operator-Valued Extension of Free Probability

Let $\mathcal{B} \subset \mathcal{A}$. A linear map

$$E : \mathcal{A} \to \mathcal{B}$$

is a **conditional expectation** if

$$E[b] = b \qquad \forall b \in \mathcal{B}$$

and

$$E[b_1 a b_2] = b_1 E[a] b_2 \qquad \forall a \in \mathcal{A}, \quad \forall b_1, b_2 \in \mathcal{B}$$

An **operator-valued probability space** consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$

Example: $M_2(\mathbb{C})$-valued probability space

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Put

$$M_2(\mathcal{A}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathcal{A} \right\}$$

and consider $\psi := \mathrm{tr} \otimes \varphi$ and $E := \mathrm{id} \otimes \varphi$, i.e.:

$$\psi \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \frac{1}{2}(\varphi(a) + \varphi(d)), \qquad E \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{pmatrix}$$

- $(M_2(\mathcal{A}), \psi)$ is a non-commutative probability space, and
- $(M_2(\mathcal{A}), E)$ is an $M_2(\mathbb{C})$-valued probability space

Consider an operator-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$. The **operator-valued distribution** of $a \in \mathcal{A}$ is given by all operator-valued moments

$$E[ab_1ab_2 \cdots b_{n-1}a] \in \mathcal{B} \qquad (n \in \mathbb{N}, b_1, \dots, b_{n-1} \in \mathcal{B})$$

Consider an operator-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$. The **operator-valued distribution** of $a \in \mathcal{A}$ is given by all operator-valued moments

$$E[ab_1 ab_2 \cdots b_{n-1} a] \in \mathcal{B} \qquad (n \in \mathbb{N}, b_1, \dots, b_{n-1} \in \mathcal{B})$$

Random variables $x_i \in \mathcal{A}$ $(i \in I)$ are **free with respect to E** (or **free with amalgamation over $\mathcal{B}$)** if

$$E[a_1 \cdots a_n] = 0$$

whenever $a_i \in \mathcal{B}\langle x_{j(i)} \rangle$ are polynomials in some $x_{j(i)}$ with coefficients from $\mathcal{B}$ and

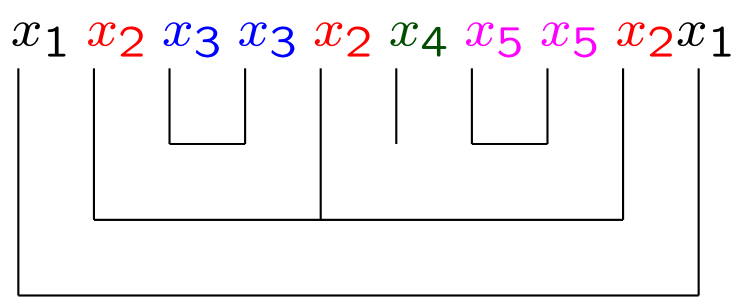
$$E[a_i] = 0 \quad \forall i \qquad \text{and} \qquad j(1) \neq j(2) \neq \cdots \neq j(n).$$

Note: polynomials in x with coefficients from $\mathcal{B}$ are of the form

- x^2
- $b_0 x^2$
- $b_1 x b_2 x b_3$
- $b_1 x b_2 x b_3 + b_4 x b_5 x b_6 + \cdots$
- etc.

b's and x do not commute in general!

Operator-valued freeness works mostly like ordinary freeness, one only has to take care of the order of the variables; in all expressions they have to appear in their original order!
Still one has factorizations of all non-crossing moments in free variables.



$$E[x_1 x_2 x_3 x_3 x_2 x_4 x_5 x_5 x_2 x_1]$$

$$= E\Big[x_1 \cdot E\big[x_2 \cdot E[x_3 x_3] \cdot x_2 \cdot E[x_4] \cdot E[x_5 x_5] \cdot x_2\big] \cdot x_1\Big]$$

For "crossing" moments one has analogous formulas as in scalar-valued case, modulo respecting the order of the variables ...

The formula

$$\varphi(x_1x_2x_1x_2) = \varphi(x_1x_1)\varphi(x_2)\varphi(x_2) + \varphi(x_1)\varphi(x_1)\varphi(x_2x_2) \\ - \varphi(x_1)\varphi(x_2)\varphi(x_1)\varphi(x_2)$$

has now to be written as

$$E[x_1x_2x_1x_2] = E\Big[x_1E[x_2]x_1\Big] \cdot E[x_2] + E[x_1] \cdot E\Big[x_2E[x_1]x_2\Big] \\ - E[x_1]E[x_2]E[x_1]E[x_2]$$

Freeness and Matrices

Easy, but crucial fact: Freeness is compatible with going over to matrices

If $\{a_1, b_1, c_1, d_1\}$ and $\{a_2, b_2, c_2, d_2\}$ are free in $(\mathcal{A}, \varphi)$, then

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

are

- in general, not free in $(M_2(\mathcal{A}), \mathrm{tr} \otimes \varphi)$
- but free with amalgamation over $M_2(\mathbb{C})$ in $(M_2(\mathcal{A}), \mathrm{id} \otimes \varphi)$

Example

Let $\{a_1, b_1, c_1, d_1\}$ and $\{a_2, b_2, c_2, d_2\}$ be free in $(\mathcal{A}, \varphi)$, consider

$$X_1 := \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \text{and} \qquad X_2 := \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

Then

$$X_1 X_2 = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

and

$$\begin{aligned} \psi(X_1 X_2) &= \Big(\varphi(a_1)\varphi(a_2) + \varphi(b_1)\varphi(c_2) + \varphi(c_1)\varphi(b_2) + \varphi(d_1)\varphi(d_2)\Big)/2 \\ &\neq (\varphi(a_1) + \varphi(d_1))(\varphi(a_2) + \varphi(d_2))/4 \\ &= \psi(X_1) \cdot \psi(X_2) \end{aligned}$$

but

$$E(X_1 X_2) = E(X_1) \cdot E(X_2)$$

Consider $E : \mathcal{A} \to \mathcal{B}$.

Define **free cumulants**

$$\kappa_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}$$

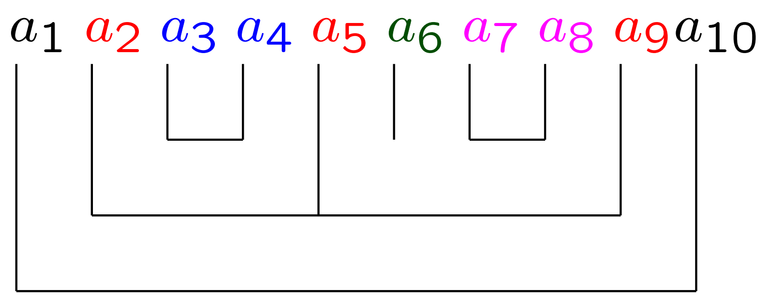
by

$$E[a_1 \cdots a_n] = \sum_{\pi \in NC(n)} \kappa_\pi^{\mathcal{B}}[a_1, \dots, a_n]$$

- arguments of $\kappa_\pi^{\mathcal{B}}$ are distributed according to blocks of π
- but now: cumulants are nested inside each other according to nesting of blocks of π

Example:

$$\pi = \Big\{\{1,10\},\{2,5,9\},\{3,4\},\{6\},\{7,8\}\Big\} \in NC(10),$$



$$\kappa_\pi^{\mathcal{B}}[a_1,\ldots,a_{10}]$$

$$= \kappa_2^{\mathcal{B}}\Big(a_1 \cdot \kappa_3^{\mathcal{B}}\big(a_2 \cdot \kappa_2^{\mathcal{B}}(a_3,a_4), a_5 \cdot \kappa_1^{\mathcal{B}}(a_6) \cdot \kappa_2^{\mathcal{B}}(a_7,a_8), a_9\big), a_{10}\Big)$$

For $a \in \mathcal{A}$ define its **operator-valued Cauchy transform**

$$G_a(b) := E[\frac{1}{b-a}] = \sum_{n \geq 0} E[b^{-1}(ab^{-1})^n]$$

and **operator-valued R-transform**

$$\begin{aligned} R_a(b) :&= \sum_{n \geq 0} \kappa_{n+1}^{\mathcal{B}}(ab, ab, \ldots, ab, a) \\ &= \kappa_1^{\mathcal{B}}(a) + \kappa_2^{\mathcal{B}}(ab, a) + \kappa_3^{\mathcal{B}}(ab, ab, a) + \cdots \end{aligned}$$

Then

$$bG(b) = 1 + R(G(b)) \cdot G(b) \qquad \text{or} \qquad G(b) = \frac{1}{b - R(G(b))}$$

On a Formal Power Series Level: Same Results as in Scalar-Valued Case

If x and y are free over $\mathcal{B}$, then

- mixed $\mathcal{B}$-valued cumulants in x and y vanish
- $R_{x+y}(b) = R_x(b) + R_y(b)$
- we have the subordination $G_{x+y}(z) = G_x(\omega(z))$

Theorem (Belinschi, Mai, Speicher 2017): Let x and y be selfadjoint operator-valued random variables free over B. Then there exists a Fréchet analytic map $\omega\colon \mathbb{H}^+(B) \to \mathbb{H}^+(B)$ so that

$$G_{x+y}(b) = G_x(\omega(b)) \text{ for all } b \in \mathbb{H}^+(B).$$

Moreover, if $b \in \mathbb{H}^+(B)$, then $\omega(b)$ is the unique fixed point of the map

$$f_b\colon \mathbb{H}^+(B) \to \mathbb{H}^+(B), \qquad f_b(w) = h_y(h_x(w) + b) + b,$$

and

$$\omega(b) = \lim_{n\to\infty} f_b^{\circ n}(w) \qquad \text{for any } w \in \mathbb{H}^+(B).$$

where

$$\mathbb{H}^+(B) := \{b \in B \mid (b - b^*)/(2i) > 0\}, \qquad h(b) := \frac{1}{G(b)} - b$$

Back to the Problem of Polynomials of Independent Random Matrices and Polynomials in Free Variables

If the random matrices $X_1, \dots, X_k$ are asymptotically freely independent, then the distribution of a polynomial $p(X_1, \dots, X_k)$ is asymptotically given by the distribution of $p(x_1, \dots, x_k)$, where

- $x_1, \dots, x_k$ are freely independent variables, and
- the distribution of x_i is the asymptotic distribution of X_i

Problem: How do we deal with a polynomial p in free variables?

Idea: Linearize the polynomial and use operator-valued convolution for the linearization $\hat{p}$!

The linearization of $p = xy + yx + x^2$ is given by

$$\hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$

This means that the Cauchy transform $G_p(z)$ is given as the (1,1)-entry of the operator-valued (3×3 matrix) Cauchy transform of $\hat{p}$:

$$G_{\hat{p}}(b) = \text{id} \otimes \varphi \left[(b - \hat{p})^{-1} \right] = \begin{pmatrix} G_p(z) & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \qquad \text{for} \quad b = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

But

$$\widehat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix} = \widehat{x} + \widehat{y}$$

with

$$\widehat{x} = \begin{pmatrix} 0 & x & \frac{x}{2} \\ x & 0 & 0 \\ \frac{x}{2} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \widehat{y} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & -1 \\ y & -1 & 0 \end{pmatrix}.$$

So $\widehat{p}$ is just the sum of two operator-valued variables

$$\widehat{p} = \begin{pmatrix} 0 & x & \frac{x}{2} \\ x & 0 & 0 \\ \frac{x}{2} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & -1 \\ y & -1 & 0 \end{pmatrix}$$

- where we understand the operator-valued distributions of $\widehat{x}$ and of $\widehat{y}$

- **and $\hat{x}$ and $\hat{y}$ are operator-valued freely independent!**

So we can use operator-valued free convolution to calculate the operator-valued Cauchy transform of $\widehat{x} + \widehat{y}$.

So we can use operator-valued free convolution to calculate the operator-valued Cauchy transform of $\widehat{p} = \widehat{x} + \widehat{y}$ in the subordination form

$$G_{\widehat{p}}(b) = G_{\widehat{x}}(\omega(b)),$$

where $\omega(b)$ is the unique fixed point in the upper half plane $\mathbb{H}_{+}(M_3(\mathbb{C})$ of the iteration

$$w \mapsto G_{\widehat{y}}(b + G_{\widehat{x}}(w)^{-1} - w)^{-1} - (G_{\widehat{x}}(w)^{-1} - w)$$

Input: $\boldsymbol{p(x,y),\ G_x(z), G_y(z)}$

$\downarrow$

Linearize $p(x,y)$ to $\widehat{p} = \widehat{x} + \widehat{y}$

$\downarrow$

$G_{\widehat{x}}(b)$ out of $G_x(z)$ and $G_{\widehat{y}}(b)$ out of $G_y(z)$

$\downarrow$

Get $w(b)$ as the fixed point of the iteration
$w \mapsto G_{\widehat{y}}(b + G_{\widehat{x}}(w)^{-1} - w)^{-1} - (G_{\widehat{x}}(w)^{-1} - w)$

$\downarrow$

$G_{\widehat{p}}(b) = G_{\widehat{x}}(\omega(b))$

$\downarrow$

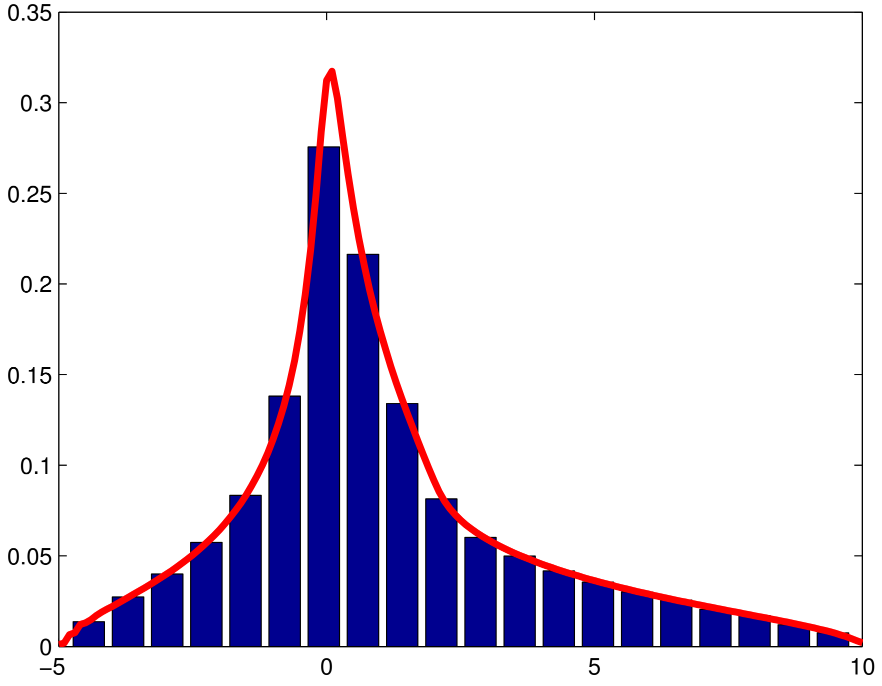
Recover $\boldsymbol{G_p(z)}$ as one entry of $\boldsymbol{G_{\hat{p}}(b)}$

Example: $p(x, y) = xy + yx + x^2$

p has linearization

$$\hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$

$P(X, Y) = XY + YX + X^2$
for independent X, Y; X is Wigner and Y is Wishart



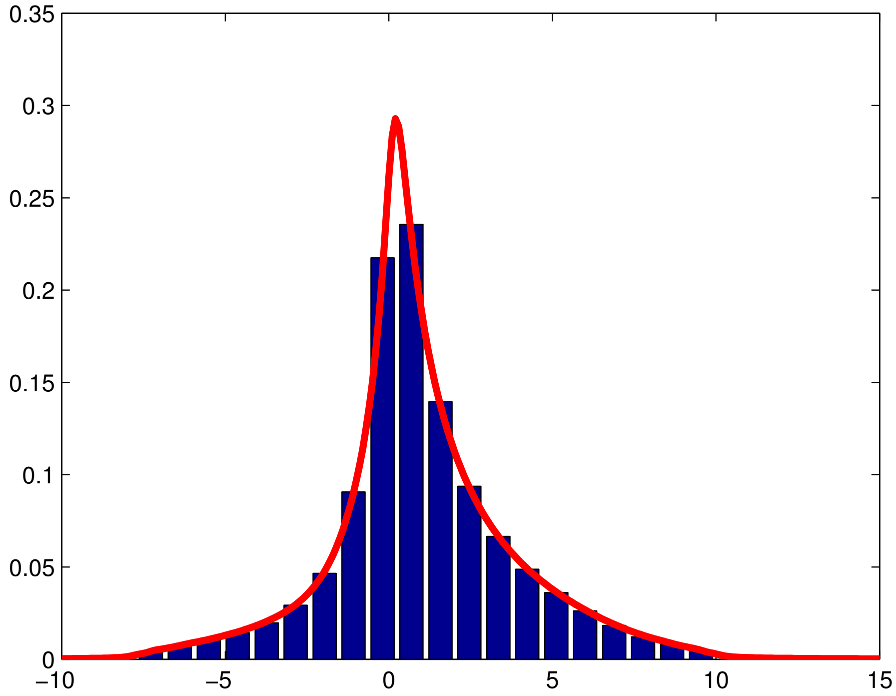
$p(x, y) = xy + yx + x^2$
for free x, y; x is semicircular and y is Marchenko-Pastur

Example: $p(x_1, x_2, x_3) = x_1 x_2 x_1 + x_2 x_3 x_2 + x_3 x_1 x_3$

p has linearization

$$\hat{p} = \begin{pmatrix} 0 & 0 & x_1 & 0 & x_2 & 0 & x_3 \\ 0 & x_2 & -1 & 0 & 0 & 0 & 0 \\ x_1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & -1 & 0 & 0 \\ x_2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & -1 \\ x_3 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$P(X_1, X_2, X_3) = X_1 X_2 X_1 + X_2 X_3 X_2 + X_3 X_1 X_3$
for independent X_1, X_2, X_3; X_1, X_2 Wigner, X_3 Wishart



$p(x_1, x_2, x_3) = x_1 x_2 x_1 + x_2 x_3 x_2 + x_3 x_1 x_3$
for free x_1, x_2, x_3; x_1, x_2 semicircular, x_3 Marchenko-Pastur

Theorem (Belinschi, Mai, Speicher 2017): Combining the selfadjoint linearization trick with our analysis of operator-valued free convolution we can provide an efficient and analytically controllable algorithm for calculating the asymptotic eigenvalue distribution of

any selfadjoint polynomial

(even rational function)

in asymptotically free random matrices.