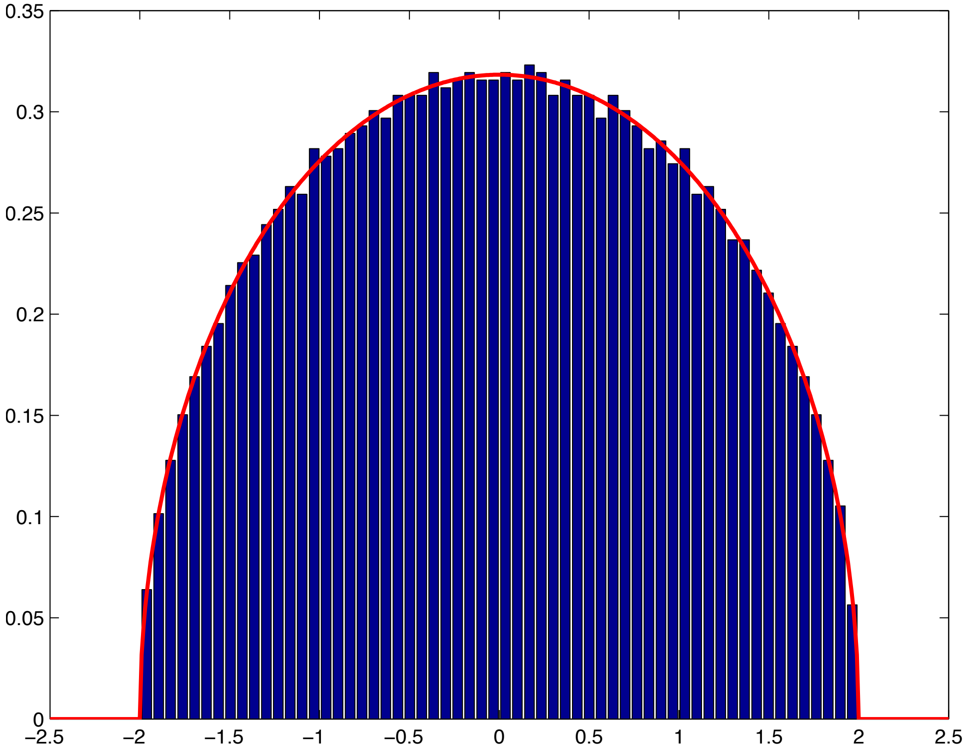
Free Stochastic Analysis

Roland Speicher
Saarland University
Saarbrücken, Germany

Wigner's semicircle law

Consider selfadjoint Gaussian $N \times N$ random matrix.



... one realization ...

N=4000

A **free Brownian motion** is given by a family $\big(S(t)\big)_{t\geq 0} \subset (\mathcal{A}, \varphi)$ of random variables ($\mathcal{A}$ von Neumann algebra, φ faithful trace), such that

- $S(0) = 0$
- each increment $S(t) - S(s)$ $(s < t)$ is semicircular with mean $= 0$ and variance $= t - s$, i.e.,

$$d\mu_{S(t)-S(s)}(x) = \frac{1}{2\pi(t-s)}\sqrt{4(t-s) - x^2}dx$$

- disjoint increments are free: for $0 < t_1 < t_2 < \cdots < t_n$,

$$S(t_1), \quad S(t_2) - S(t_1), \quad \ldots, \quad S(t_n) - S(t_{n-1}) \qquad \text{are free}$$

A **free Brownian motion** is given

- abstractly, by a family $\big(S(t)\big)_{t\geq 0}$ of random variables with
  - $S(0) = 0$
  - each $S(t) - S(s)$ $(s < t)$ is $(0, t-s)$-semicircular
  - disjoint increments are free
- asymptotically, as the limit of matrix-valued (Dyson) Brownian motions
- concretely, by the sum of creation and annihilation operators on the full Fock space

Free Brownian motions as matrix limits

Let $(X_N(t))_{t\geq 0}$ be a symmetric $N \times N$-matrix-valued Brownian motion, i.e.,

$$X_N(t) = \begin{pmatrix} B_{11}(t) & \dots & B_{1N}(t) \\ \vdots & \ddots & \vdots \\ B_{N1}(t) & \dots & B_{NN}(t) \end{pmatrix}, \qquad \text{where}$$

- B_{ij} are, for $i \geq j$, independent classical Brownian motions

- $B_{ij}(t) = B_{ji}(t)$.

Then, $\boldsymbol{(X_N(t))_{t\geq 0}} \xrightarrow{\text{distr}} \boldsymbol{(S(t))_{t\geq 0}}$, in the sense that almost surely

$$\lim_{N\to\infty} \text{tr}\Big(X_N(t_1)\cdots X_N(t_n)\Big) = \varphi\Big(S(t_1)\cdots S(t_n)\Big) \qquad \forall 0 \leq t_1, t_2, \dots, t_n$$

Intermezzo on realisations on Fock spaces

Classical Brownian motion can be realized quite canonically by operators on the symmetric Fock space.

Similarly, free Brownian motion can be realized quite canonically by operators on the full Fock space

First: symmetric Fock space ...

For Hilbert space $\mathcal{H}$ put

$$\mathcal{F}_s(\mathcal{H}) := \bigoplus_{n\geq 0}^{\infty} \mathcal{H}^{\otimes \mathrm{sym}\, n}, \qquad \text{where} \qquad \mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$$

with inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_{\mathrm{sym}} = \delta_{nm} \sum_{\pi \in S_n} \prod_{i=1}^{n} \langle f_i, g_{\pi(i)} \rangle.$$

Define creation and annihilation operators

$$a^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n$$

$$a(f) f_1 \otimes \cdots \otimes f_n = \sum_{i=1}^{n} \langle f, f_i \rangle f_1 \otimes \cdots \otimes \check{f}_i \otimes \cdots \otimes f_n$$

$$a(f)\Omega = 0$$

... and classical Brownian motion

Put $\qquad \varphi(\cdot) := \langle \Omega, \cdot \Omega \rangle, \qquad x(f) := a(f) + a^*(f)$

then $(x(f))_{f \in \mathcal{H}, f \text{ real}}$ is Gaussian family with covariance

$$\varphi\Big(x(f)x(g)\Big) = \langle f, g \rangle.$$

In particular, choose $\mathcal{H} := L^2(\mathbb{R}_+)$, $f_t := 1_{[0,t)}$, then

$B_t := a(1_{[0,t)}) + a^*(1_{[0,t)})$ is classical Brownian motion,

meaning

$$\varphi(B_{t_1} \cdots B_{t_n}) = E[W_{t_1} \cdots W_{t_n}] \qquad \forall 0 \leq t_1, \dots, t_n$$

Now: full Fock space ...

For Hilbert space $\mathcal{H}$ put

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n\geq 0}^{\infty} \mathcal{H}^{\otimes n}, \qquad \text{where} \qquad \mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$$

with usual inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle = \delta_{nm} \langle f_1, g_1 \rangle \cdots \langle f_n, g_n \rangle.$$

Define creation and annihilation operators

$$b^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n$$
$$b(f) f_1 \otimes \cdots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n$$
$$b(f)\Omega = 0$$

... and free Brownian motion

Put $\qquad \varphi(\cdot) := \langle \Omega, \cdot \Omega \rangle, \qquad x(f) := b(f) + b^*(f)$

then $(x(f))_{f \in \mathcal{H}, f \text{ real}}$ is semicircular family with covariance

$$\varphi\Big(x(f)x(g)\Big) = \langle f, g \rangle.$$

In particular, choose $\mathcal{H} := L^2(\mathbb{R}_+)$, $f_t := 1_{[0,t)}$, then

$S_t := b(1_{[0,t)}) + b^*(1_{[0,t)})$ is free Brownian motion.

Semicircle as real part of one-sided shift

Consider case of one-dimensional $\mathcal{H}$. Then this reduces to

$$\text{basis:} \quad \Omega = e_0, \quad e_1, \quad e_2, \quad e_3, \quad \dots$$

and one-sided shift l

$$le_n = e_{n+1}, \qquad l^* e_n = \begin{cases} e_{n-1}, & n \geq 1 \\ 0, & n = 0 \end{cases}$$

One-sided shift is canonical non-unitary isometry

$$l^* l = 1, \qquad l^* l \neq 1 \quad (= 1 - \text{projection on } \Omega)$$

With $\varphi(a) := \langle \Omega, a\Omega \rangle$ we claim: distribution of $l + l^*$ is semicircle.

Moments of $l + l^*$

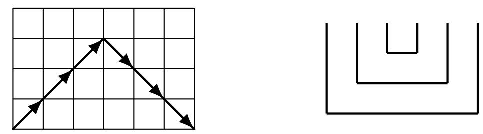
In the calculation of $\langle \Omega, (l + l^*)^n \Omega \rangle$ only such products in creation and annihilation contribute, where we never annihilate the vacuum, and where we start and end at the vacuum. So odd moments are zero.

Examples

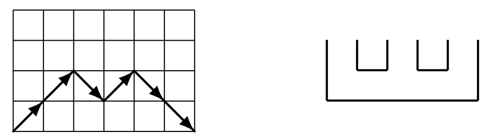
$$\varphi\Big((l+l^*)^2\Big): \quad l^*l$$
$$\varphi\Big((l+l^*)^4\Big): \quad l^*l^*ll, \quad l^*ll^*l$$
$$\varphi\Big((l+l^*)^6\Big): \quad l^*l^*l^*lll, \quad l^*ll^*l^*ll, \quad l^*ll^*ll^*l, \quad l^*l^*ll^*ll, \quad l^*l^*lll^*l$$

Those contributing terms are in clear bijection with non-crossing pairings (or with Dyck paths).

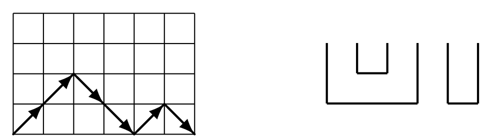
(l^*, l^*, l^*, l, l, l)



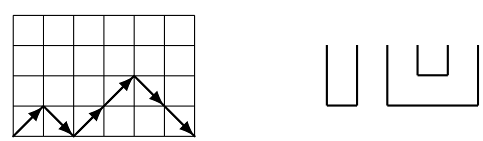
(l^*, l^*, l, l^*, l, l)



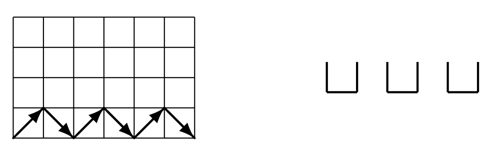
(l^*, l, l^*, l^*, l, l)



(l^*, l^*, l, l, l^*, l)



(l^*, l, l^*, l, l^*, l)



Free Stochastic Calculus

- Kümmerer, Speicher: Stochastic integration on the Cuntz algebra O_∞, 1992
- **Biane, Speicher: Stochastic calculus with respect to free Brownian motion and analysis on Wigner space, 1998**
- Deya, Schott: On the rough-paths approach to non-commutative stochastic calculus, 2013
- Kargin On free stochastic differential equations, 2011
- Schluechtermann, Wibmer: Numerical Solution of Free Stochastic Differential Equations, 2023
- Niua, Wei, Yina, Zeng: Stochastic theta methods for free stochastic differential equations, 2024
- Jekel, Kemp, Nikitopoulos: A Martingale Approach to Non-commutative Stochastic Calculus, 2023

Free Stochastic Calculus

Goal: For processes

$$\Big(A(t)\Big)_{t\geq 0}, \quad \Big(B(t)\Big)_{t\geq 0} \quad \subset \mathcal{A}$$

(functions of the free Brownian motion) we want to define a stochastic integral of the form

$$\int A(t)dS(t)B(t).$$

Ito–Type Definition for Adapted Processes

As usual: processes must be adapted

$\Big(A(t)\Big)_{t\geq 0}$ is **adapted** if

$$A(t) \in \mathsf{vN}\Big(S(\tau) \mid \tau \leq t\Big) \qquad \forall t \geq 0$$

Then define for piecewise constant processes

$$\int A(t)dS(t)B(t) := \sum_i A(t_i)\Big(S(t_{i+1}) - S(t_i)\Big)B(t_i)$$

and extend by continuity

Norm Estimates for Free Stochastic Integrals

- **Ito isometry:** for the L^2 norm $\|a\|_2^2 := \varphi(aa^*)$ we have

$$\| \int A(t) dS(t) B(t) \|_2^2 = \int \|A(t)\|_2^2 \cdot \|B(t)\|_2^2 dt$$

  note: this is essentially the fact that for a semicircle S of variance dt, which is free from $\{a, a^*, b, b^*\}$ we have

$$\varphi(aSbb^*Sa^*) = \varphi(bb^*)\varphi(aa^*)dt$$

- **free Burkholder-Gundy inequality for $p = \infty$:** for the operator norm we have the much deeper estimate

$$\| \int A(t) dS(t) B(t) \|^2 \leq c \cdot \int \|A(t)\|^2 \cdot \|B(t)\|^2 dt$$

Free Ito Formula

We have as for classical Brownian motion

$$dS(t)dS(t) = dt$$

Free Ito Formula

We have as for classical Brownian motion

$$dS(t)dS(t) = dt$$

... but that's not all, we also need

$$dS(t)AdS(t) = \varphi(A)dt$$

for A adapted

[Classically we have of course: $dW(t)AdW(t) = Adt$]

Stochastic Analysis on "Wigner" space

Starting from a free Brownian motion $\left(S(t)\right)_{t \geq 0}$ we define multiple "Wigner" integrals

$$I(f) = \int \cdots \int f(t_1, \ldots, t_n) dS(t_1) \ldots dS(t_n)$$

for scalar-valued functions $f \in L^2(\mathbb{R}_+^n)$, by avoiding the diagonals, i.e. we understand this as

$$I(f) = \int \cdots \int_{\text{all } t_i \text{ distinct}} f(t_1, \ldots, t_n) dS(t_1) \ldots dS(t_n)$$

Definition of Wigner integrals

More precisely: for f of form

$$f = 1_{[s_1,t_1]\times\cdots\times[s_n,t_n]}$$

for pairwisely disjoint intervals $[s_1, t_1], \dots, [s_n, t_n]$ we put

$$I(f) := (S_{t_1} - S_{s_1}) \cdots (S_{t_n} - S_{s_n})$$

Extend $I(\cdot)$ linearly over set of all off-diagonal step functions (which is dense in $L^2(\mathbb{R}_+^n)$. Then observe Ito-isometry

$$\varphi[I(g)^* I(f)] = \langle f, g \rangle_{L^2(\mathbb{R}_+^n)}$$

and extend I to closure of off-diagonal step functions, i.e., to $L^2(\mathbb{R}_+^n)$.

Note: free stochastic integrals are usually bounded operators

Free Haagerup Inequality [Bozejko 1991; Biane, Speicher 1998]:

$$\left\| \int \cdots \int f(t_1, \ldots, t_n) dS(t_1) \ldots dS(t_n) \right\| \leq (n+1) \|f\|_{L^2(\mathbb{R}_+^n)}$$

Intermezzo: combinatorics and norms

Consider free semicirculars $s_1, s_2, \dots$ of variance 1. Since $(s_1 + \cdots + s_n)/\sqrt{n}$ is again a semicircular element of variance 1 (and thus of norm 2), we have

$$\left\| \left(\frac{s_1 + \cdots + s_n}{\sqrt{n}} \right)^k \right\| = 2^k$$

The free Haagerup inequality says that this is drastically reduced if we subtract the diagonals, i.e.,

$$\lim_{n\to\infty} \left\| \frac{1}{n^{k/2}} \sum_{\substack{i(1),\dots,\imath(k)=1 \\ \text{all } i(.) \text{ different}}}^{n} s_{i(1)} \cdots s_{i(k)} \right\| = k+1$$

Intermezzo: combinatorics and norms

Note: one can calculate norms from asymptotic knowledge of moments!

If x is selfadjoint and φ faithful (as our φ for the free Brownian motion is) then one has

$$\|x\| = \lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty} \sqrt[p]{\varphi(|x|^p)} = \lim_{m\to\infty} \sqrt[2m]{\varphi(x^{2m})}$$

Intermezzo: combinatorics and norms

Note: one can calculate norms from asymptotic knowledge of moments!

If x is selfadjoint and φ faithful (as our φ for the free Brownian motion is) then one has

$$\|x\| = \lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty} \sqrt[p]{\varphi(|x|^p)} = \lim_{m\to\infty} \sqrt[2m]{\varphi(x^{2m})}$$

So, if s is a semicircular element, then

$$\varphi(s^{2m}) = c_m = \frac{1}{1+m}\binom{2m}{m} \sim 4^m,$$

thus

$$\|s\| = \lim_{m\to\infty} \sqrt[2m]{c_m} \sim \sqrt[2m]{4^m} = 2$$

Exercise: combinatorics and norms

Consider free semicirculars $s_1, s_2, \ldots$ of variance 1. Prove by considering moments that

$$\left\| \frac{1}{n} \sum_{i,j=1}^{n} s_i s_j \right\| = 4$$

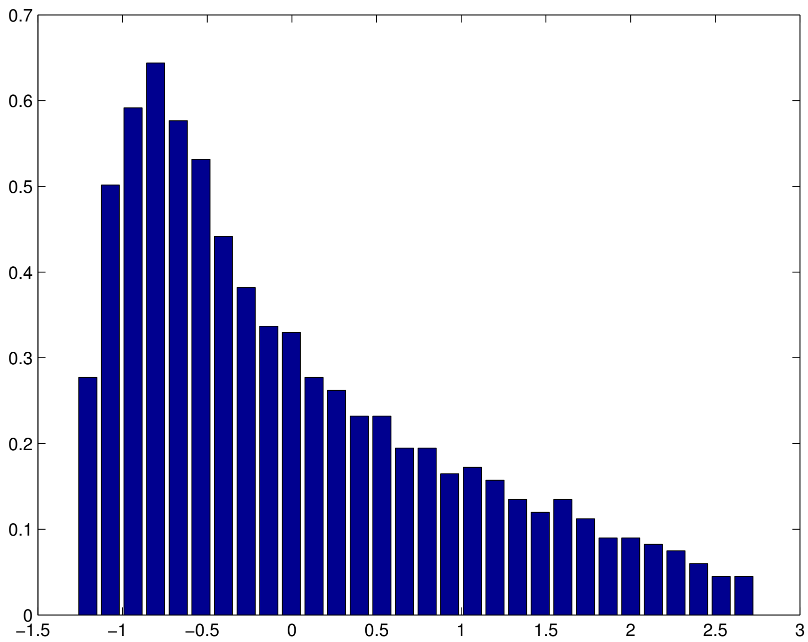
but

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{i,j=1, i \neq j}^{n} s_i s_j \right\| = 3$$

2000 eigenvalues of the matrix

$$\frac{1}{12} \sum_{ij=1, i \neq j}^{12} X_i X_j,$$

where the X_i are independent Gaussian random matrices



Multiplication of Multiple Wigner Integrals

The Ito formula shows up in multiplicaton of two multiple Wigner integrals

$$\int f(t_1)dS(t_1) \cdot \int g(t_2)dS(t_2)$$

$$= \iint f(t_1)g(t_2)dS(t_1)dS(t_2) +$$

Multiplication of Multiple Wigner Integrals

The Ito formula shows up in multiplicaton of two multiple Wigner integrals

$$\int f(t_1)dS(t_1) \cdot \int g(t_2)dS(t_2)$$

$$= \iint f(t_1)g(t_2)dS(t_1)dS(t_2) + \int f(t)g(t) \underbrace{dS(t)dS(t)}_{dt}$$

Multiplication of Multiple Wigner Integrals

The Ito formula shows up in multiplicaton of two multiple Wigner integrals

$$\int f(t_1)dS(t_1) \cdot \int g(t_2)dS(t_2)$$

$$= \iint f(t_1)g(t_2)dS(t_1)dS(t_2) + \int f(t)g(t)\underbrace{dS(t)dS(t)}_{dt}$$

$$= \iint f(t_1)g(t_2)dS(t_1)dS(t_2) + \int f(t)g(t)dt$$

Multiplication of Multiple Wigner Integrals

$$\iint f(t_1,t_2)dS(t_1)dS(t_2)\cdot\int g(t_3)dS(t_3)$$

$$=\iiint f(t_1,t_2)g(t_3)dS(t_1)dS(t_2)dS(t_3)$$

$$+\iint f(t_1,t)g(t)dS(t_1)\underbrace{dS(t)dS(t)}_{dt}$$

$$+\iint f(t,t_2)g(t)\underbrace{dS(t)dS(t_2)dS(t)}$$

Multiplication of Multiple Wigner Integrals

$$\iint f(t_1,t_2)dS(t_1)dS(t_2)\cdot\int g(t_3)dS(t_3)$$

$$= \iiint f(t_1,t_2)g(t_3)dS(t_1)dS(t_2)dS(t_3)$$

$$+ \iint f(t_1,t)g(t)dS(t_1)\underbrace{dS(t)dS(t)}_{dt}$$

$$+ \iint f(t,t_2)g(t)\underbrace{dS(t)dS(t_2)dS(t)}_{dt\varphi[dS(t_2)]=0}$$

Multiplication of Multiple Wigner Integrals

Consider $f \in L^2(\mathbb{R}_+^n)$, $\qquad g \in L^2(\mathbb{R}_+^m)$

For $0 \le p \le \min(n, m)$, define

$$f \overset{p}{\frown} g \in L^2(\mathbb{R}_+^{n+m-2p})$$

by

$$f \overset{p}{\frown} g(t_1, \dots, t_{m+n-2p})$$

$$= \int f(t_1, \dots, t_{n-p}, s_1, \dots, s_p) g(s_p, \dots, s_1, t_{n-p+1}, \dots, t_{n+m-2p}) ds_1 \cdots ds_p$$

Then we have

$$I(f) \cdot I(g) = \sum_{p=0}^{\min(n,m)} I(f \overset{p}{\frown} g)$$

Example: $f \in L^2(\mathbb{R}_+^3)$, $g \in L^2(\mathbb{R}_+^4)$

$$\int f(t_1, t_2, t_3) dS(t_1) dS(t_2) dS(t_3) \cdot \int g(s_1, s_2, s_3, s_4) dS(s_1) \dots dS(s_4)$$

$$= \bigcirc \bigcirc \bigcirc \mid \bigcirc \bigcirc \bigcirc \bigcirc$$

$$= \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc + \bigcirc \bigcirc \circ \circ \bigcirc \bigcirc \bigcirc$$

$$+ \bigcirc \circ \circ \circ \circ \bigcirc \bigcirc + \circ \circ \circ \circ \circ \circ \bigcirc$$

$$\int 1_{[0,1]}(t)dS(t) = S(1) - S(0) = S \qquad \text{semicircular variable}$$

What is

$$U_n := \int 1_{[0,1]^n}(t_1, \ldots, t_n) dS(t_1) \cdots dS(t_n)$$

We have

$$S \cdot U_n = \bigcirc \mid \bigcirc \bigcirc \cdots \bigcirc = \bigcirc \bigcirc \cdots \bigcirc + \circ \circ \bigcirc \bigcirc \cdots \bigcirc$$

$$= U_{n+1} + U_{n-1}$$

Thus

$$S \cdot U_n = U_{n+1} + U_{n-1} \qquad \text{recursion for Chebycheff polynomials}$$

$$U_1 = S, \quad U_2 = S^2 - 1, \qquad U_3 = S^3 - 2S, \qquad \ldots$$

Note

$$U_n = S^n + \text{smaller degree polynomial}$$

In this special case, Haagerup inequality is saying that

$$\|S^n\| = 2^n$$

is reduced to

$$\|U_n\| = n + 1$$

For example,

$$\|S\| = 2, \qquad \|S^2\| = 4, \qquad \|S^3\| = 8$$
$$\|S\| = 2, \qquad \|S^2 - 1\| = 3, \qquad \|S^3 - 2S\| = 4$$

Note

$$U_n = S^n + \text{smaller degree polynomial}$$

In this special case, Haagerup inequality is saying that

$$\|S^n\| = 2^n$$

is reduced to

$$\|U_n\| = n + 1$$

For example,

$$\|S\| = 2, \qquad \|S^2\| = 4, \qquad \|S^3\| = 8$$
$$\|S\| = 2, \qquad \|S^2 - 1\| = 3, \qquad \|S^3 - 2S\| = 4$$

This follows here from

$$\|U_n\| = \sup_{|t|\leq 2} |U_n(t)| \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

Compare to classical analogue

$$\int 1_{[0,1]}(t)dB(t) = B(1) - B(0) = N \qquad \text{normal variable}$$

$$H_n := \int 1_{[0,1]^n}(t_1, \dots, t_n) dB(t_1) \cdots dB(t_n)$$

We have

$$N \cdot H_n = \bigcirc \mid \bigcirc \bigcirc \cdots \bigcirc$$

$$= \bigcirc \bigcirc \bigcirc \cdots \bigcirc + \circ \circ \bigcirc \cdots \bigcirc + \circ \bigcirc \circ \cdots \bigcirc + \circ \bigcirc \bigcirc \cdots \circ$$

$$= H_{n+1} + nH_{n-1}$$

$$N \cdot H_n = H_{n+1} + nH_{n-1} \qquad \text{recursion for Hermite polynomials}$$

$$H_1 = N, \qquad H_2 = N^2 - 1, \qquad H_3 = N^3 - 3N, \qquad \dots$$

Note that for $n \geq 1$:

$$\varphi[I(f)] = \int f(t_1, \ldots, t_n) \varphi[dS(t_1) \cdots dS(t_n)] = 0$$

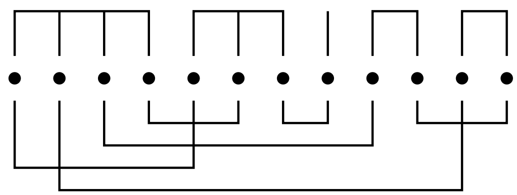
Thus, for $f_i \in L^2(\mathbb{R}_+^{n_i})$

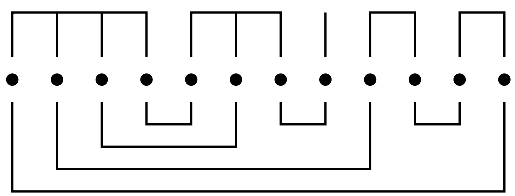
$$\varphi[I(f_1) \cdots I(f_r)] = \text{only terms with total contractions}$$

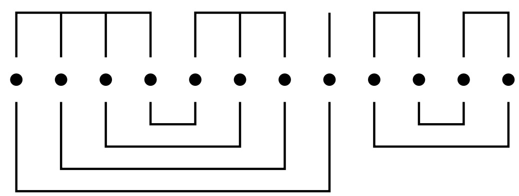
$$= \sum_{\pi \in NC_2(n_1 \otimes \cdots \otimes n_r)} \int_\pi f_1 \otimes \cdots \otimes f_r$$

Example: $I(f_1)I(f_2)I(f_3)I(f_4)I(f_5)$ with

$$(n_1, n_2, n_3, n_4, n_5) = (4, 3, 1, 2, 2)$$







The first picture contributes in the classical case, but not in the free case. The other two pictures contribute here.

More general: all $\pi \in NC(n_1 + \cdots + n_r)$ contribute with

$$\pi \wedge \{[n_1], [n_1, n_1 + n_2], \dots\} = 0$$

In particular, we have for $f, g \in L^2(\mathbb{R}_+^n)$

$$
\begin{aligned}
&\varphi[I(f)I(g)^*] \\
&= \int f(t_1, \dots, f_n)\bar{g}(s_1, \dots, s_n)\varphi[dS(t_1)\dots dS(t_n) \cdot dS(s_n)\dots dS(s_1)] \\
&= \int f(t_1, \dots, t_n)\bar{g}(t_1, \dots, t_n)dt_1 \dots dt_n \\
&= \langle f, g \rangle_{L^2(\mathbb{R}_+^n)}
\end{aligned}
$$

and for $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ with $n \neq m$

$$\varphi[I(g)I(g)^*] = 0$$

or more general, for $f, g \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+^n)$

$$\varphi[I(f)I(g)^*] = \langle f, g \rangle$$

Free Chaos Decomposition

One has the canonical isomorphism

$$L^2(\{S(t) \mid t \geq 0\}) \,\hat{=}\, \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}_+^n), \qquad f \,\hat{=}\, \bigoplus_{n=0}^{\infty} f_n,$$

via

$$f = \sum_{n=0}^{\infty} I(f_n) = \sum_{n=0}^{\infty} \int \cdots \int f_n(t_1, \dots, t_n) dS(t_1) \dots dS(t_n).$$

The set

$$\{I(f_n) \mid f_n \in L^2(\mathbb{R}_+^n)\}$$

is called **n-th (Wigner) chaos**.

Note: Polynomials in free semicircular elements

$$p(s_1, \ldots, s_m)$$

can be realized as elements from **finite chaos**

$$\{I(f) \mid f \in \bigoplus_{\text{finite}} L^2(\mathbb{R}_+^n)\}$$

Understanding the Distributions of Stochastic Integrals

We would like to understand the distributions of selfadjoint elements from fixed (or finite, or even more general) chaos.

- Can we distinguish distributions of $I(f)$ and $I(g)$ from different chaos?

- Do we understand the regularity of distributions of
$$\int f(t_1,\dots,t_n)dS(t_1)\cdots dS(t_n),$$
or of solutions of free stochastic differential equations like
$$dX_t = f(X_t)dS_t g(X_t)?$$

Theorem (Kemp, Nourdin, Peccati, Speicher 2012): Consider, for fixed n, a sequence $f_1, f_2, \dots \in L^2(\mathbb{R}_+^n)$ with $f_k^* = f_k$ and $\|f_k\|_2 = 1$ for all $k \in \mathbb{N}$. Then the following statements are equivalent.

(i) We have $\lim_{k\to\infty} \varphi[I(f_k)^4] = 2$.

(ii) We have for all $p = 1, 2, \dots, n-1$ that

$$\lim_{k\to\infty} f_k \overset{p}{\frown} f_k = 0 \qquad \text{in } L^2(\mathbb{R}_+^{2n-2p}).$$

(iii) The selfadjoint variable $I(f_k)$ converges in distribution to a semicircular variable of variance 2.

Corollary: For $n \geq 2$ and $f \in L^2(\mathbb{R}_+^n)$, the law of $I(f)$ is not semicircular

Thus for $n \neq 2$

$$\{\text{distributions in first chaos}\} \cap \{\text{distribution in } n\text{-th chaos}\} = \emptyset$$

The more general question for $n \neq m$

$\{$distributions in m-th chaos$\}\cap\{$distribution in n-th chaos$\}$ =???

is still open.

[For the classical case one knows that all Wiener chaoses have disjoint distributions.]

The Regularity Questions

We would like to understand distributions of functions of free random variables, say

- polynomials in finitely many free semicirculars $s_1, \dots, s_n$:

$$p(s_1, \dots, s_n)$$

- "polynomials" in infinitely many free semicirculars $\{dS(t) \mid t \geq 0\}$:

$$\int f(t_1, \dots, t_n) dS(t_1) \cdots dS(t_n)$$

In particular, we would like to know whether these distributions have atoms or have a (regular) density

A Disappointing Insight

At the moment our algorithm for calculating the distribution of polynomials in free variables does not allow to make such qualitative statements.

Our distributions arise essentially as the solutions of matrix-valued fixpoint equations. But we have at the moment no tools to derive qualitative features of the solution of such equations.

But there are other promising routes to attack these questions …

Some Answers on the Absence of Atoms

Theorem (Shlyakhtenko, Skoufranis 2013; Mai, Speicher, Weber 2014:)
Let p be a non-constant selfadjoint polynomial and $s_1, \dots, s_n$ free semicirculars. Then the distribution of $p(s_1, \dots, s_n)$ does not have atoms.

Theorem (Mai 2015):
The distribution of a non-constant finite selfadjoint Wigner integral

$$\sum_{k=1}^{n} \int f_k(t_1, \dots, t_k) dS(t_1,) \cdots dS(t_k)$$

does not have atoms.

Idea of Proof

The results of Mai, Speicher, Weber rely on having a calculus of non-commutative derivatives for our polynomials:

- having atoms for some polynomials implies by differentiation that one also has atoms for the derivative

- but then, by iteration, one should have atoms for linear polynomials

- which is not the case

The result of Mai on stochastic integrals relies on a version of such a differential calculus in the setting of stochastic integrals ...

... this is the free Malliavin calculus

Free Malliavin Calculus (Biane, Speicher 1998)

The **free gradient operator** ∇ is given by:

$$\nabla_t \Big(\int f(t_1, \dots, t_n) dS(t_1) \cdots dS(t_n) \Big)$$

$$= \sum_{k=1}^{n} \int f(t_1, \dots, t_{k-1}, t, t_{k+1}, \dots, t_n)$$

$$dS(t_1) \cdots dS(t_{k-1}) \otimes dS(t_{k+1}) \dots dS(t_n)$$

Free Malliavin Calculus (Biane, Speicher 1998)

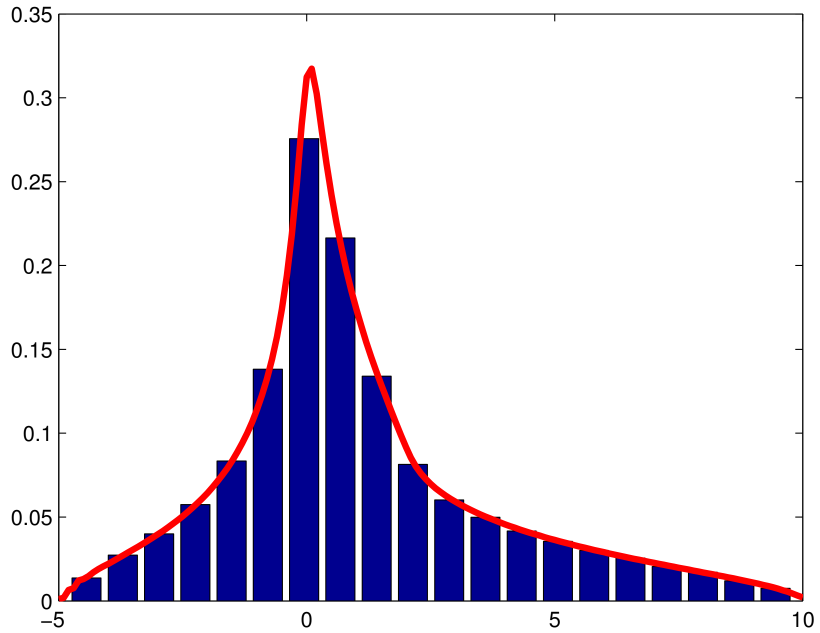
The adjoint of the free gradient is the **free divergence operator**:

$$\delta\left(\int f_t(t_1,\dots,t_n,s_1,\dots,s_m)dS(t_1)\cdots dS(t_n)\otimes dS(s_1)\cdots dS(s_m)\right)$$
$$=\int f_t(t_1,\dots,t_n,s_1,\dots,s_m)dS(t_1)\cdots dS(t_m)dS(t)dS(s_1)\cdots dS(s_m)$$

The free divergence $\delta=\nabla^*$ is the **free Skorohod integral**.

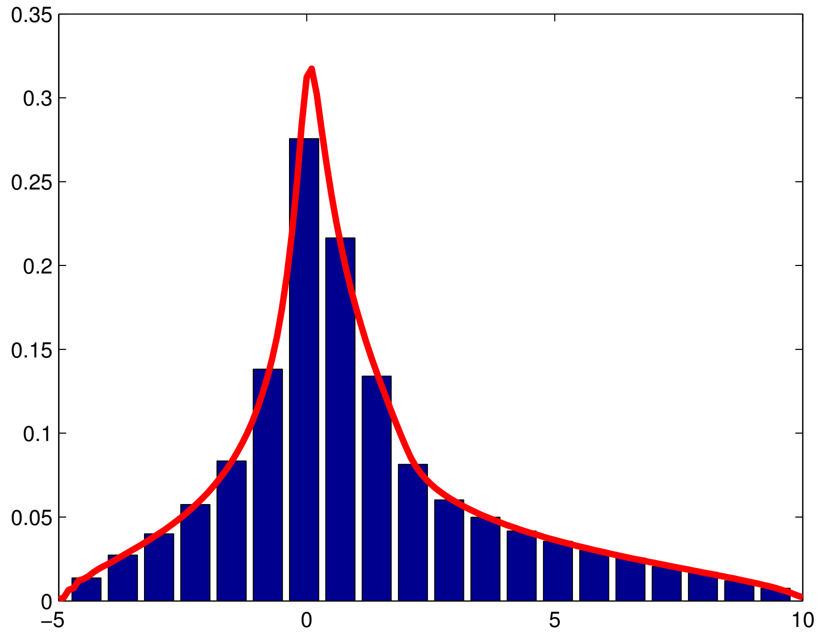
Applied to adapted processes it reduces to the free Ito integral

Consider the polynomial $p(x,y) = xy + yx + x^2$ in a semicircular x and a Marchenko-Pastur y which are free.



We can prove that the distribution of $p(x,y)$ has no atoms!

Consider the polynomial $p(x, y) = xy + yx + x^2$ in a semicircular x and a Marchenko-Pastur y which are free.



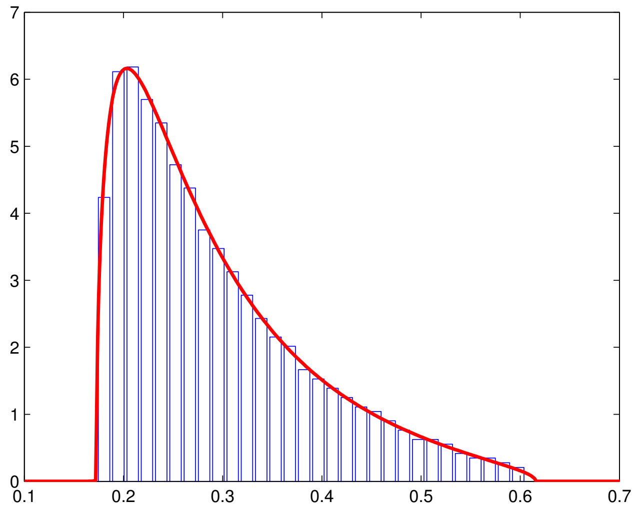
We can prove that the distribution of $p(x, y)$ has no atoms!

But how about rational functions?

Consider the rational function

$$r = (4-x_1)^{-1} + (4-x_1)^{-1} x_2 \left[(4 - x_1) - x_2 (4 - x_1)^{-1} x_2 \right]^{-1} x_2 (4-x_1)^{-1}$$

in two free semicirculars x_1 and x_2.



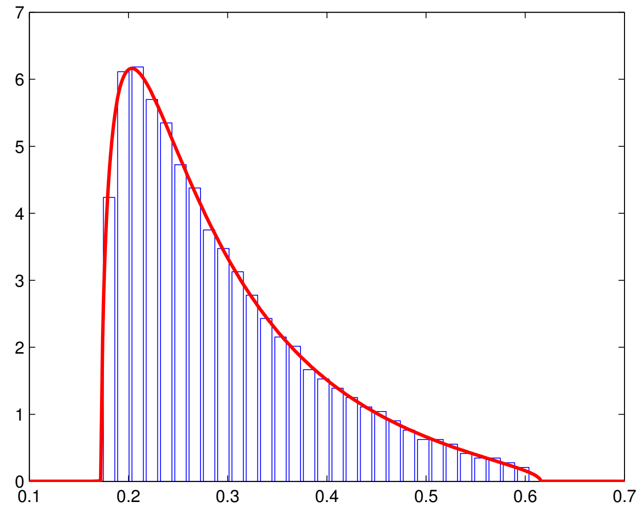
From our algorithm (linearization and operator-valued convolution) we see that its distribution has no atoms.

But can we prove it?

Consider the rational function

$$r = (4-x_1)^{-1}+(4-x_1)^{-1}x_2\left[(4-x_1)-x_2(4-x_1)^{-1}x_2\right]^{-1}x_2(4-x_1)^{-1}$$

in two free semicirculars x_1 and x_2.



From our algorithm (linearization and operator-valued convolution) we see that its distribution has no atoms.
But can we prove it? — **Indeed, we can!**

Theorem (Mai, Speicher, Yin 2022):

For any non-constant selfadjoint non-commutative rational function r, its evaluation in free semicirculars has no atom in its distribution.

Theorem (Mai, Speicher, Yin 2022):

For any non-constant selfadjoint non-commutative rational function r, its evaluation in free semicirculars has no atom in its distribution.

Theorem (Mai, Speicher, Yin 2022):

For any non-constant selfadjoint non-commutative rational function r, its evaluation in free semicirculars has no atom in its distribution.

Thank you for your attention!

Theorem (Mai, Speicher, Yin 2022):

For any non-constant selfadjoint non-commutative rational function r, its evaluation in free semicirculars has no atom in its distribution.

Thank you for your attention!