Approximating integrals with discontinuous integrands driven by multifractional Brownian motion

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Introduction

We consider equidistant approximations of stochastic integrals driven by multifractional Brownian motion with discontinuous integrands. Specifically, we establish the rate of convergence for equidistant approximations of pathwise stochastic integrals:

$$\int_0^1 \Psi'(X_s) dX_s \approx \sum_{k=1}^{n} \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right),$$

where $t_k = \frac{k}{n}$, $k = 0, 1, \ldots, n$. Here, Ψ is a convex function, and X denotes a multifractional Brownian motion. The integral is interpreted as a pathwise Stieltjes integral, following the integration theory for discontinuous integrands developed in [Chen et al.(2019)] using a modification of Zähle's fractional integration theory [Zähle(1998), Zähle(2001)].

Zhe Chen, Lasse Leskelä, and Lauri Viitasaari.
Pathwise Stieltjes integrals of discontinuously evaluated stochastic processes.
Stochastic Process. Appl., 129(8):2723–2757, 2019.

A similar problem was addressed in [Azmoodeh et al.(2024)] for the case when the driving process X is centered, Gaussian and Hölder continuous of order $H > \frac{1}{2}$. Additionally, in [Azmoodeh et al.(2024)], X satisfies the following conditions:

- its variance function $V(t)$ is non-decreasing on $[0, 1]$, $V(1) = 1$,
- and its variogram function is represented as

$$\mathbb{E}(X_t - X_s)^2 = \sigma^2 |t - s|^{2H} + o\left(|t - s|^{2H}\right), \quad \text{as } |t - s| \to 0.$$

Examples of such processes include fractional, bifractional and sub-fractional Brownian motions, the fractional Ornstein–Uhlenbeck process, and normalized multi-mixed fractional Brownian motion, among others.

Ehsan Azmoodeh, Pauliina Ilmonen, Nourhan Shafik, Tommi Sottinen, and Lauri Viitasaari.
On sharp rate of convergence for discretization of integrals driven by fractional Brownian motions and related processes with discontinuous integrands.
J. Theoret. Probab., 37(1):721–743, 2024.

In [Azmoodeh et al.(2024)], the exact rate of convergence for approximations of stochastic integrals in the L^1-distance is found to be proportional to n^{1-2H}, which corresponds to the known rate in the case of smooth integrands.

We focus on approximating integrals driven by multifractional Brownian motion. This process generalizes fractional Brownian motion by allowing the Hurst index to vary over time. In this case, the variance function of the process is

$$V(t) = t^{2H_t},$$

which is generally non-monotone. We establish a rate of convergence proportional to n^{1-2H} with $H = \min\{\min_t H_t, \alpha\}$, where α is a Hölder exponent of H_t.

Multifractional Brownian motion: Definition and examples

Let $H\colon [0,1] \to (\frac{1}{2}, 1)$ be a continuous function satisfying the following assumptions:

(A1) $H_{\min} := \min_{t\in[0,1]} H_t > \frac{1}{2}$ and $H_{\max} := \max_{t\in[0,1]} H_t < 1$.

(A2) There exist constants $C > 0$ and $\alpha \in (\frac{1}{2}, 1]$ such that for all $t, s \in [0,1]$

$$|H_t - H_s| \le C\,|t - s|^{\alpha}.$$

There exist several generalizations of fractional Brownian motion to the case where the Hurst index H is varying with time.

Example 1 (Moving-average multifractional Brownian motion [Peltier and Lévy Véhel(1995)])

Multifractional Brownian motion was first introduced by [Peltier and Lévy Véhel(1995)]. Their definition is based on the Mandelbrot–van Ness representation for fractional Brownian motion. The moving-average multifractional Brownian motion is defined by

$$X_t = C_1\left(H_t\right) \int_{-\infty}^{t} \left[(t-s)_+^{H_t - \frac{1}{2}} - (-s)_+^{H_t - \frac{1}{2}} \right] dW_s, \qquad (1)$$

where $W = \{W_t, t \in \mathbb{R}\}$ is a two-sided Wiener process, $x_+ = \max\{x, 0\}$, and

$$C_1(H) = \left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)} \right)^{1/2} = \frac{(2H\Gamma(2H)\sin(\pi H))^{1/2}}{\Gamma\left(H+\frac{1}{2}\right)}.$$

Romain-François Peltier and Jacques Lévy Véhel.
Multifractional Brownian motion: definition and preliminary results.
INRIA research report, 2645, 1995.

Example 2 (Multifractional Volterra-type Brownian motion [R. and Shevchenko(2010)])

The next definition of a multifractional Brownian motion is based on the integral representation of the fractional Brownian motion through a Brownian motion on a finite interval developed in [Norros et al.(1999)]. The multifractional Volterra-type Brownian motion is the process

$$X_t = \int_0^t K_{H_t}(t,s)\,dW_s, \tag{2}$$

where $W = \{W_t, t \geq 0\}$ is a Wiener process, and $K_H(t,s)$ is the Molchan kernel defined by

$$K_H(t,s) = C_2(H) s^{\frac{1}{2}-H} \int_s^t (v-s)^{H-\frac{3}{2}} v^{H-\frac{1}{2}}\,dv, \quad H \in (\tfrac{1}{2}, 1),$$

with $C_2(H) = C_1(H)(H - \frac{1}{2})$.

K. Ralchenko and G. Shevchenko. Path properties of multifractal Brownian motion. *Theory Probab. Math. Statist.*, 80:119–130, 2010.

Example 3 (Harmonizable multifractional Brownian motion [Benassi et al.(1997)])

The harmonizable multifractional Brownian motion is defined by

$$X_t = C_3(H_t) \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{\frac{1}{2}+H_t}} W(dx), \tag{3}$$

where $C_3(H) = (H\Gamma(2H)\sin(\pi H)/\pi)^{1/2}$, and $W(\cdot)$ is a complex random measure on $\mathbb{R}$ such that

1) for all $A, B \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{E}W(A)\overline{W(B)} = \lambda(A \cap B),$$

where λ is the Lebesgue measure;

2) for an arbitrary sequence $\{A_1, A_2, \dots\} \subset \mathcal{B}(\mathbb{R})$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, we have

$$W\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i\geq 1} W(A_i),$$

(here $\{W(A_i), i \geq 1\}$ are centered normal random variables);

3) for all $A \in \mathcal{B}(\mathbb{R})$,

$$W(A) = \overline{W(-A)},$$

4) for all $\theta \in \mathbb{R}$,

$$\left\{e^{i\theta} W(A), A \in \mathcal{B}(\mathbb{R})\right\} \stackrel{d}{=} \{W(A), A \in \mathcal{B}(\mathbb{R})\}.$$

Albert Benassi, Stéphane Jaffard, and Daniel Roux.
Elliptic Gaussian random processes.
Rev. Mat. Iberoamericana, 13(1):19–90, 1997.

In the sequel, we consider a generalization of the fractional Brownian motion defined by $X_t = B_t^{H_t}$, $t \in [0,1]$, where $\left\{B_t^H, t \in [0,1], H \in \left(\frac{1}{2}, 1\right)\right\}$ is a family of random variables such that

(B1) for a fixed $H \in \left(\frac{1}{2}, 1\right)$, the process $\left\{B_t^H, t \in [0,1]\right\}$ is a fractional Brownian motion with the Hurst parameter H;

(B2) for all $t \in [0,1]$ and all $H_1, H_2 \in [H_{\min}, H_{\max}]$,

$$\mathbb{E}\left(B_t^{H_1} - B_t^{H_2}\right)^2 \leq C(H_1 - H_2)^2, \tag{4}$$

where C is a constant that may depend on $H_{\min}$ and $H_{\max}$.

The above conditions are satisfied, for instance, by every one of the generalizations described in Examples 1–3, since conditions (B1) and (B2) hold for representations (1)–(3), see [Peltier and Lévy Véhel(1995), R. and Shevchenko(2010), Cohen(1999)] respectively.

For further reference, we collect necessary properties of the variance and variogram functions of multifractional Brownian motion in the following lemma.

Lemma 4

The multifractional Brownian motion $X = \{X_t, t \in [0,1]\}$ has the following properties.

(i) *For all $t \in [0,1]$*

$$V(t) := \mathbb{E}X_t^2 = t^{2H_t}.$$

(ii) *For all $t, s \in [0,1]$*

$$\mathbb{E}\left(X_t - X_s\right)^2 \le |t-s|^{2H_{\min}} + C\,|t-s|^{H_{\min}+\alpha} + C\,|t-s|^{2\alpha}.$$

For a convex function Ψ, let Ψ' denote its one sided derivative. In condition (A1) we assume that the function H_t is bounded away from one. This guarantees that

$$\int_0^1 \frac{1}{\sqrt{V(s)}}\, ds \leq \int_0^1 s^{-H_{\max}}\, ds < \infty.$$

Then by [Chen et al.(2019)] $\int_0^1 \Psi'(X_s) dX_s$ exists as a pathwise Riemann–Stieltjes integral; moreover, it satisfies the following chain rule:

$$\int_0^1 \Psi'(X_s) dX_s = \Psi\left(X_1\right) - \Psi\left(X_0\right). \tag{5}$$

Let $t_k = \frac{k}{n}$, $k = 0, 1, \ldots, n$, be an equidistant partition of the interval $[0, 1]$. Throughout the article we use the notation

$$\varphi(a) := \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}, \quad a \in \mathbb{R}. \tag{6}$$

Main result

Theorem 5

Let $X_t = B_t^{H_t}$ be a multifractional Brownian motion with the Hurst function H_t satisfying (A1)–(A2). Let Ψ be a convex function with the left-sided derivative Ψ', and let μ denote the measure associated with the second derivative of Ψ such that $\int_{\mathbb{R}} \varphi(a)\mu(da) < \infty$. Then for any $\widetilde{H} \in (\frac{1}{2}, H_{\min}] \cap (\frac{1}{2}, \alpha)$,

$$\mathbb{E}\left|\int_0^1 \Psi'(X_s)dX_s - \sum_{k=1}^{n} \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right)\right|$$

$$\leq \int_{\mathbb{R}} \int_0^1 s^{-H_s} \varphi\left(\frac{a}{s^{H_s}}\right) ds\, \mu(da) \left(\frac{1}{n}\right)^{2\widetilde{H}-1} + \int_{\mathbb{R}} R_n(a)\mu(da), \quad (7)$$

where the remainder satisfies

$$\int_{\mathbb{R}} R_n(a)\mu(da) \leq C n^{-\min\{2\widetilde{H}-H_{\max}, H_{\min}+\alpha-1, 2\alpha-1\}}. \quad (8)$$

Remark 1

Assumption $\widetilde{H} \in (\frac{1}{2}, H_{\min}] \cap (\frac{1}{2}, \alpha)$ guarantees that that the remainder is negligible compared to the first term in (7). Indeed, we have

$$2\widetilde{H} - H_{\max} > 2\widetilde{H} - 1, \ H_{\min} + \alpha - 1 > 2\widetilde{H} - 1, \text{ and } 2\alpha - 1 > 2\widetilde{H} - 1.$$

Hence,

$$\frac{\int_{\mathbb{R}} R_n(a)\mu(da)}{n^{1-2\widetilde{H}}} \to 0, \quad \text{as } n \to \infty.$$

Remark 2

One can formulate the statement of Theorem 5 more precisely by considering the cases $\alpha > H_{\min}$ and $\alpha \in (\frac{1}{2}, H_{\min}]$ separately. Evidently, in the case $\alpha > H_{\min}$, (7) holds with $\widetilde{H} = H_{\min}$. And in the general case, i.e., $\alpha > \frac{1}{2}$, one has

$$\mathbb{E}\left|\int_0^1 \Psi'(X_s)dX_s - \sum_{k=1}^{n} \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right)\right| \leq Cn^{1-2\min\{H_{\min},\alpha\}}. \quad (9)$$

Note that for $\frac{1}{2} < \alpha \leq H_{\min}$, the leading term in (7) has the same order $n^{1-2\alpha}$ as the remainder; so we cannot obtain more precise rate of convergence than (9).

Remark 3

When the function H is sufficiently smooth and the difference between $H_{\max}$ and $H_{\min}$ is rather small, one can establish a lower bound in addition to (7). Namely, under additional assumptions

$$\alpha > H_{\max} \quad \text{and} \quad 3H_{\max} - 2H_{\min} < 1, \tag{10}$$

the following inequality holds

$$\mathbb{E}\left| \int_0^1 \Psi'(X_s) dX_s - \sum_{k=1}^{n} \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right) \right| \\ \geq \int_{\mathbb{R}} \int_0^1 s^{-H_s} \varphi\left(\frac{a}{s^{H_s}}\right) ds\, \mu(da) \left(\frac{1}{n}\right)^{2H_{\max}-1} + \int_{\mathbb{R}} R_n(a)\mu(da), \tag{11}$$

where the same remainder that satisfies (8). Due to assumptions (10) the remainder in (11) is negligible compared to the leading term.

Remark 4

In particular, the assumptions (10) hold in the case $H_t = H = \text{const}$ (i.e., when X is a fractional Brownian motion). Indeed, in this case one can take $\alpha = 1$, $H_{\min} = H_{\max} = H$, and the bounds (7) and (11) imply that

$$\mathbb{E}\left|\int_0^1 \Psi'(X_s)dX_s - \sum_{k=1}^{n} \Psi'\left(X_{t_{k-1}}\right)\left(X_{t_k} - X_{t_{k-1}}\right)\right| \\ = \int_{\mathbb{R}} \int_0^1 s^{-H} \varphi\left(\frac{a}{s^H}\right) ds\, \mu(da) \left(\frac{1}{n}\right)^{2H-1} + \widetilde{R}_n(a),$$

with $\widetilde{R}_n(a) \leq Cn^{-H}$. This coincides with the result of [Azmoodeh et al.(2024)] for the case of fractional Brownian motion.

Moreover, since in the case $H_t = H = \text{const}$ we have an exact rate of convergence n^{1-2H}, we see that the result of Theorem 5 cannot be improved substantially.

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