Introduction.

Multi-scale system, 2 scales

man dog

man	virus
planet	man
climate	weather

separation of time scale $\varepsilon \sim 0$

$\vdash\!\!\dashv$ 1 $\qquad\qquad$ $\vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashv$ $1/\varepsilon$

$x^\varepsilon_t \qquad y_{t/\varepsilon} \equiv y^\varepsilon_t$

Example $\dot{x} = y$, $\dot{y} = -x$ $\qquad t \mapsto x^2(t) + y^2(t)$ is invariant

perturbation: $\frac{d}{dt}\left(\cdot \right) = \begin{pmatrix} y^\varepsilon \\ -x^\varepsilon \end{pmatrix} + \varepsilon V(x, y)$

$I^\varepsilon_t = \frac{1}{2}\left(x^\varepsilon(\tfrac{t}{\varepsilon})\right)^2 + \frac{1}{2}\left(y^\varepsilon(\tfrac{t}{\varepsilon})\right)^2, \qquad x = I\cos\theta,\ y = I\sin\theta$

$\frac{d}{dt} I^\varepsilon_t = \varepsilon I^\varepsilon_t\, V\left(I^\varepsilon_t \cos\theta(t/\varepsilon) + I^\varepsilon_t \sin\theta(t/\varepsilon)\right).$

Hamiltonian system on R^{2n}, $z = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad \dot{z}^\varepsilon(t) = J\nabla f(z^\varepsilon(t)) + \varepsilon V(z(t))$

$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

z^ε is closed to z. on $[0,1]$

If $0 = \{f, g\} = \langle \nabla f, J\nabla g \rangle$, $\qquad g(z^\varepsilon(t)) \sim \varepsilon$. on $[0,1]$.

$g(z^\varepsilon(t/\varepsilon))$ is observable.

$z^\varepsilon(t) = (\underbrace{g(z^\varepsilon(t))}_{\text{slow}}, \underbrace{\theta(z^\varepsilon(t))}_{\text{fast}})$

Freidlin–Frentzell

Random perturbation: $\dot{z}^\varepsilon_t = J\nabla f(z^\varepsilon) + \varepsilon \dot{W}(t)$ in R^2

Completely integrable Hamiltonian system.

Suppose $\exists$ d poisson commuting functions on M^{2d}

Consider $\sum_{i=1}^{d} H_i W_t^i$

$$dZ_t = \frac{1}{\varepsilon} \sum_{i=1}^{d} H_i(Z_t) \circ dW_t^i + V(Z_t)dt.$$

Fast oscillations

perodic

Suppose the fast motion is periodic or ergodic, We may observe its influence on the flow variable, despite of they are not tractible.

seasonal flu.

Riemann Lebesque lemma

If $f \in L^1(\mathbb{R},\mathbb{R})$, $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) e^{inx} dx = 0$

$\int f(x) \cos(nx) dx \xrightarrow[(n\to\infty)]{} 0$

Consider $f: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ bdd, periodic in its second argument, uniformly lipschitz in first argument.

$$\frac{d}{dt} x_t^\varepsilon = f(x_t^\varepsilon, \frac{t}{\varepsilon})$$

Then $x_t^\varepsilon \to \bar{x}_t$, $\dot{\bar{x}}_t = \bar{f}(x_t)$, $\bar{f}(x) = \frac{1}{L}\int_0^L f(s)ds$.

Stationary process Birkhoff ergodic theorem.

Suppose (Y_t) is stationary, $(Y_{s+\cdot}) \overset{law}{=} (Y_\cdot)$, and ergodic ($\mathbb{P} = \mathcal{L}(Y_\cdot)$ is invariant under θ, any function invariant under θ is $\mathbb{P}$-a.s. constant)

Then for any $f \in L^1$, $\frac{1}{t}\int_0^t f(Y_s)ds \xrightarrow[a.e.]{} \int f(y)\pi(dy) = \bar{f}$.

$\int_0^2 f(Y_s)ds = \int_0^1 f(Y_{s+1})ds$, $\frac{1}{n}\sum_{i=1}^{n} f \circ T^n \to \bar{f}$

Limit Theorems

① $\int_0^t f(Y_{r/\varepsilon})\,dr \longrightarrow \int f(z)\,\pi(dz)$

② $Y \perp\!\!\!\perp W$, $\quad W \in \mathbb{R}^m$, $\quad F : \mathbb{R}^m \longrightarrow \mathbb{R}^d$ bdd
nice

$\mathbb{E}\left(\int_0^t F(Y_{r/\varepsilon})\,dW_r \,\middle|\, Y\right)$ is a BM.

$\mathbb{E}\left(\int_0^t F(Y_{r/\varepsilon})\,dW_r \,\middle|\, Y\right)^2$ Variance

$= \int_0^t F(Y_{r/\varepsilon})\left(F(Y_{r/\varepsilon})\right)^T dr$

$\longrightarrow \int F(z)\left(F(z)\right)^T \pi(dz) = \Sigma\Sigma^T$

So it converges to a BM : $\Sigma W(t)$.

- Averaging principle

$$\dot{x}_t^\varepsilon = f(x_t^\varepsilon, Y_{t/\varepsilon})$$

- Stochastic Averaging principle.

$$dx_t^\varepsilon = f(x_t, Y_{t/\varepsilon})\,d\tilde{W}_t + g(x_t, Y_{t/\varepsilon})\,dW_t$$

$\mathcal{A}^y = (f f^T)_{i,j}(x,y)\dfrac{\partial^2}{\partial x_i \partial x_j} + g_k(x,y)\dfrac{\partial}{\partial x_k}$

$\bar{\mathcal{A}} = \int \mathcal{A}^y\,\pi(dy)$. $\qquad$ Martingale problem tightness

$$dy_r = V_0(y_r)\,dr + \sum V_k(y_r)\,dW_r$$

— Suppose that $\bar{f}=0$, $\int_0^t f(Y_{r/\varepsilon})\,dr \xrightarrow[(\varepsilon\to 0)]{} 0$.

$$\varepsilon^{-1/2}\int_0^t f(Y_{r/\varepsilon})\,dr = \sqrt{\varepsilon}\int_0^{t/\varepsilon} f(Y_{r'})\,dr' \qquad Y_{r/\varepsilon} = y_r^\varepsilon$$

$$\dot{x}_t^\varepsilon = \sqrt{\varepsilon}^{-1} f(x_t^\varepsilon, Y_{t/\varepsilon})$$

Suppose $\mathcal{L}h = f$ has a C^2 solution, $\mathcal{L}$ generator for (Y_t).

e.g. Y is elliptic, reversible w.r.t. π

or on compact manifolds

Then
$$\frac{1}{\varepsilon}\int_{s/\varepsilon}^{t/\varepsilon} f(Y_r)\,dr = h(Y_{t/\varepsilon}) - h(Y_{s/\varepsilon}) - \frac{1}{\varepsilon}\int_{s/\varepsilon}^{t/\varepsilon} dh(Y_{r\varepsilon})\,d\{Y_r\}$$

$$\frac{1}{\sqrt{\varepsilon}}\int_{s/\varepsilon}^{t/\varepsilon} f(Y_r)\,dr = \sqrt{\varepsilon}\left(h(Y_{t/\varepsilon}) - h(Y_{s/\varepsilon})\right) - \underbrace{\int_{s/\varepsilon}^{t/\varepsilon} dh(Y_r)\,V_k(Y_r)\,dW_r}_{\text{martingale.}}$$

converges to a martingale.

$$\varepsilon\,\mathbb{E}\left(\int_0^{t/\varepsilon} f(Y_r)\,dr\right)^2$$
$$= 2\varepsilon\int_0^t\int_0^s \mathbb{E}\left(f(Y_0)\,f(Y_{\frac{s-r}{\varepsilon}})\right)dr\,ds$$
$$= t\int_0^{r/\varepsilon} \mathbb{E}\left(f(Y_0)\,f(Y_u)\right)du$$
$$\longrightarrow t\int_0^\infty \mathbb{E}\left(f(Y_0)\,f(Y_u)\right)du$$
$$= t\left\langle f, \int_0^\infty P_u f\,du\right\rangle = t\int f\,\mathcal{L}^{-1} f\,d\pi$$

$$\int_0^t\int_0^s f(s-r)\,dr\,ds$$
$$= \frac{1}{4}\int_0^r\int_{-u}^{2t-u} f(u)\,du\,dv$$
$$= \frac{t}{2}\int_0^t f(u)\,du$$

Thm (z_j) stating ergodic $\mathbb{E}\{z_{j+1} | \mathcal{F}_j\} = 0$ $X_n(t) = \frac{1}{\sqrt{n}} \sum_1^{[nt]} z_k$. 1-5
$E(z_n^2) = \sigma^2 < \infty$ $\to \sigma^2 W(t)$

(Y_n) – Markov chain $P(x, dy)$, invariant measure

$$\pi(x, dy)\,\pi(dx) = \pi(y, dx)\,\pi(dy)$$

Stationary Defin $Pf(x) = \int f(y) P(x, dy)$

Lemma. Suppose $\int v(x)\, \pi(dx) = 0$

$\int v^2(x)\, \pi(dx) < \infty$ $H = L^2(\pi)$

$$\sigma^2 = \int \frac{1+\lambda}{1-\lambda} E_v(d\lambda) < \infty$$

E_v is spectral measure $f(P) = \int_{\sigma(p)} f(\lambda)\, E(d\lambda) : H \to H$

$$\langle v, f(P) v\rangle = \int f(\lambda)\, E_v(d\lambda)$$

Recall as self-adjoint operator $E : B(\mathbb{R}) \to B(H, H)$

$E(B)$ is an orthogonal projection

$(E(B))^2 = E(B)$, $E(B_1 \cap B_2) = E(B_1) \cdot E(B_2)$

Then $$\lim_{n\to\infty} \frac{1}{n} \mathbb{E} \left| \sum_{k=1}^n v(Y_k) \right|^2 = \sigma^2$$

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} v(Y_j) \longrightarrow \sigma B \quad \text{on } D([0,T]).$$

$(I-P)u_\varepsilon = (u_\varepsilon - Pu_\varepsilon) = V - \varepsilon u_\varepsilon$ $(1+\varepsilon) u_\varepsilon - Pu_\varepsilon = V$. $\mathbb{E}\{u_\varepsilon(Y_{n+1}) - u_\varepsilon(Y_n) - \varepsilon u_\varepsilon(Y_n) + v(Y_n) | \mathcal{F}_n\} = 0$

Non-Central limit Theorem. Rosenblatt constructed

(Y_n) correlated s.t. $n^\beta \sum_{k=1}^n H_2(Y_j) \longrightarrow$ non-Gaussian measure.

$E(Y_0 Y_n) \sim n^{\alpha}$, $\alpha \in (0, \frac{1}{2})$

There are associated invariance principle.

$$M_\varepsilon^n = \sum_{j=0}^{n-1} \Big(\underbrace{u_\varepsilon(Y_{j+1}) - u_\varepsilon(Y_j)}_{-\xi_n^\varepsilon} + \overbrace{(V - \varepsilon u_\varepsilon)(Y_j)}^{\eta_n^\varepsilon} \Big) \xrightarrow{L^2} M_n$$

Stationary. each term $= u_\varepsilon(Y_1) - P u_\varepsilon(Y_0)$

$$X_n = \sum_{j=1}^n v(Y_j) = M_\varepsilon^n + \xi_n^\varepsilon + \eta_n^\varepsilon,$$

$\xi_n^\varepsilon \to 0$, η_n^ε converges

$\eta_n^\varepsilon = \ldots$

Markov stationary process.

Suppose we have a transition function $P_t(x,dy)$ which has an invariant measure π. Let Y_t be a Markov process with

$$\mathbb{P}(Y_t \in A \mid \mathcal{F}_s) = P_{t-s}(x,A).$$

and $Y_0 \sim \mu$. Then $Y_t \sim \mu$. In fact

$$\mathcal{L}(Y_{t+\cdot}) = \mathcal{L}(Y_\cdot) \quad \text{for any } t \geq 0.$$

Uniqueness of invariant measure is equivalent to law of $(Y_\cdot)$ is ergodic w.r.t. the shift θ.

Averaging principle:

$$\dot{x}_t^\varepsilon = f(x_t, Y_{t/\varepsilon})$$

By divide $[0,T]$ into small intervals, we may entertain the idea:

$$\hat{x}_t^\varepsilon = x_0 + \sum_j \underbrace{\int_{t_j}^{t_{j+1}} f(\hat{x}_{t_j}, Y_{s/\varepsilon})\, ds}$$

$$\downarrow$$

$$\sum \Delta t_j \underbrace{\int f(\hat{x}^\varepsilon_{t_j}, y)\, \pi(dy)}$$

$$\sum \Delta t_j\, \bar{f}(\tilde{x}^\varepsilon_{t_j})$$

$$\hat{x}_t^\varepsilon \sim x_t^\varepsilon$$

We expect that $x_t^\varepsilon \to \bar{x}_t$, $\quad \frac{d}{dt}\bar{x}_t = \bar{f}(\bar{x}_t)$.

Assumption: Suppose (#) $dy_t = \frac{1}{\varepsilon} V_0(x, y_t)\,dt + \frac{1}{\sqrt{\varepsilon}} \sum_{k=1}^m V_k(x, y_t) \circ dW_t^k$, y_t takes value in a compact manifold, $V_k \in BC^3$, $V_0 \in BC^2$.

$\sum_1^m \langle v, V_k(x, y)\rangle^2 \geq \lambda |v|^2$, λ independent of x, y, $v \in T_y Y$

If V depends on x, uniformly elliptic, $\int f(x,y)\,\mu^x(dy) \in BC^2$.

Lemma 1. Let $F: Y \to \mathbb{R}$ be bounded measurable

Then $\| \int_0^t (F(y_r^\varepsilon) - \bar{F})\,dr \|_p \lesssim \varepsilon^{\frac{1}{p}} t^{1-\frac{1}{p}} |F|_{\text{Lip}}$ $|F|_{\text{osc}}$

Observe constant is independent of F. $\forall\, p \geq 2$.

Proof. Firstly $\left| \int_0^t (F(y_r^\varepsilon) - \bar{F})\,dr \right| \leq 2t\, \underbrace{|F|_{\text{osc}}}_{=\sup F - \inf F}$ ①

Assume $\bar{F} = 0$ for simplicity, $|F - \bar{F}| \leq 2|F|_{\text{osc}}$.

$$\mathbb{E}\left(\int_0^t F(y_r^\varepsilon)\,dr\right)^2 = 2\int_0^t \int_0^r P_s(F\, P_{r-s} F)\,ds\,dr$$

$$\leq 4|F|_{\text{osc}} \int_0^t \int_0^r |P_{r-s} F|\,ds\,dr$$

$$\leq 4|F|_{\text{osc}}^2 \int_0^t \int_0^r e^{-\frac{c|s-r|}{\varepsilon}}\,ds\,dr$$

$$= 4|F|_{\text{osc}}^2 \cdot \varepsilon t \qquad |F|_{\text{osc}}\, \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} \qquad ②$$

Interpolate ① and ②

Lemma 2. $\| \int_s^t (b(x_u, y_r^\varepsilon) - \bar{b}(x_u))\,dr \|_p$ $u \leq s \leq t$

$$\lesssim \varepsilon^{\frac{1}{p}} |t-s|^{1-\frac{1}{p}} \overbrace{|b(x_u, \cdot)|_{\text{osc}}}^{|b|_{\text{Lip}}}$$

Apply lemma 1 to solution of SDE for y at $\Phi_{u,\cdot}(y_u)$

Simplest example

proposition Suppose that (Y_t) solves (#),

Let $b : \mathbb{R}^d \times M \to \mathbb{R}^d$ be Lipschitz and bounded

$$|b(x,y) - b(x',y')| \le K(|x-x'| + |y-y'|).$$

$\dot{x}_t^\varepsilon = b(x_t^\varepsilon, Y_{t/\varepsilon})$, Then $x_t^\varepsilon \xrightarrow{(P)} x_t$ where $\dot{\bar{x}}_t = \bar{b}(\bar{x}_t)$.

Proof.

- Firstly $\left| \int b(x,y)\,\pi(dy) - \int b(x',y)\,\pi(dy) \right| \le K|x-x'|$.

  So $\dot{\bar{x}}_t = f(\bar{x}_t)$ has a well posed global solution and

Let Δ : $\vdash\!\!\longrightarrow\!\!\dashv_T$ be a discretization.

$\Delta(\varepsilon) = o(\varepsilon)$ to be chosen, e.g. $\Delta = \varepsilon \log \frac{1}{\varepsilon}$.

s

$\varphi(s)$ is one of two end points.

- $$\sup_{t \le T} \left| \int_0^t \left(b(x_s^\varepsilon, Y_{s/\varepsilon}) - b(x^\varepsilon_{\varphi(s)}, Y_{s/\varepsilon}) \right) ds \right|$$

  $$\le K \int_0^T |x_s^\varepsilon - x^\varepsilon_{\varphi(s)}|\,ds \le KT\Delta.$$

  Lipschitz

  $$|x_s^\varepsilon - x_r^\varepsilon| \le \int_s^t |b|_\infty dr.$$

  Uniform Lipschitz (Hölder sufficient.)

$$(x_t^\varepsilon - \bar{x}_t) \doteq \sum_k \int_{t_k}^{t_{k+1}} \left(b(x_r^\varepsilon, Y_{r/\varepsilon}) - b(x^\varepsilon_{t_k}, y_{r/\varepsilon}) \right) dr$$

$$+ \sum_k \int_{t_k}^{t_{k+1}} \left(b(x^\varepsilon_{t_k}, Y_{r/2}) - \bar{b}(x^\varepsilon_{t_k}) \right) dr$$

$$+ \sum_k \int_{t_k}^{t_{k+1}} \left(\bar{b}(x^\varepsilon_{t_k}) - \bar{b}(\bar{x}_r) \right) dr$$

$$|\mathrm{III}| \le \underbrace{K \sum_k \int_{t_k}^{t_{k+1}} |x^\varepsilon_{t_k} - x^\varepsilon_r|\,dr}_{\lesssim \Delta} + K \int_0^t \underbrace{|x_r^\varepsilon - \bar{x}_r|}_{\text{linear}}\,dr$$

$$|\mathrm{I}| \le \sum_k \int_{t_k}^{t_{k+1}} K |x_r^\varepsilon - x^\varepsilon_{t_k}|\,dr \lesssim \Delta$$

$$\left\| \sum_k \int_{t_k}^{t_{k+1}} \left(b(x^\varepsilon_{t_k}, Y_{r/\varepsilon}) - \bar{b}(x^\varepsilon_{t_k}) \right) dr \right\|_2$$

$$\leq \sum_{k=1}^{N} \varepsilon^{\frac{1}{p}} \Delta^{1-\frac{1}{p}} |b - \bar{b}|_{Lip}$$

$$+ \sum_{k \neq j}$$

$$\varepsilon^{\frac{1}{p}} \left(\varepsilon \log \frac{1}{\varepsilon}\right)^{1-\frac{1}{p}} = \left(\log \frac{1}{\varepsilon}\right)^{-\frac{1}{p}} \xrightarrow[(\varepsilon \to 0)]{} 0, \quad N \sim \Delta$$

$k \neq j$

$$\mathbb{E}\left(\int_0^{t'} b(x, Y_{r/\varepsilon}) \right)^2 \qquad \bar{b} = 0$$

$$\leq 4 |H|^2_{osc} \int_0^t \int_0^r e^{-\frac{c|s-r|}{\varepsilon}} e^{-\frac{c(k-j)\Delta}{\varepsilon}} \, dr \, ds$$

$$\sim \varepsilon t \cdot e^{-(k-j)\left(\log \frac{1}{\varepsilon}\right)^2} \leq \varepsilon t \, e^{-\left(\log \frac{1}{\varepsilon}\right)^2}$$

$$\sum_{k \neq j} = \frac{1}{\Delta^2} \varepsilon \Delta \, e^{-\left(\log \frac{1}{\varepsilon}\right)^2}$$

$$= \frac{e^{-\log \frac{1}{\varepsilon} \cdot \log \frac{1}{\varepsilon}}}{\varepsilon \log \frac{1}{\varepsilon}} \to 0, \quad \Bigg| \quad e^{-\left(\log \frac{1}{\varepsilon}\right) c} = \varepsilon^{c}$$

$$e^{-\log a} = \frac{1}{a}$$

Long range dependence.

The critical point of a 2nd order phase transitions so far represents in physics the most important instance where CLT breaks down

• $$\frac{1}{n^\alpha} \sum_{i=1}^{n} X_i \not\to \text{Gaussian}$$

Donsker's invariance principle $\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{1}^{[nt]} X_i \not\to BM$

The Scale α would yield the self-similarity exponent of the process in the domain of attractions

fBM, Hermit process.

Lamperti, Embrechts, Maejima.

A mean zero stationary increment process, with self similarity (and $Y(0)=0$, $\sigma^2 = E(Y(1))^2 < \infty$).

Then $$\mathbb{E}(Y(t)\,Y(s)) = \frac{1}{2}\sigma^2 (t^{2H} + s^{2H} - |t-s|^{2H}) \quad ⊛$$

A Gaussian process with ⊛ is fBM.

$\mathbb{E}(B_{t+s} - B_s)^2 = t^{2H}$, $\quad \mathbb{E}\left(B_{t+s+1} - B_{t+s}\right)(B_{s+1} - B_s)$

$\sim t^{2H-2}$, t large

Dynamics very different.

Invariant measure is difficult

River Nile. Pharaonic peri. Hurst 1906-1968

fer-oh Nile basin

Nilometers

Mandelbrot - Van Ness

$$B_t = \int_0^t (t-u)^{H-\frac{1}{2}} dW_u + \int_{-\infty}^0 \left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right) dW_u$$

$$B_{at} = a^H B_t$$

$$B_t = \int_{-\infty}^t K(t,s)\, dW_s \qquad \text{Voltrerra process}$$

Consider $dY_t = -Y_t dt + dB_t^H$

Does $\varepsilon^{-\frac{1}{2}} \int_0^t G(Y_{s/\varepsilon})\, dB_s$ converges to W_t

- Yes if $H^*(m) < \frac{1}{2}$, where $H^*(m) = m(H-1)+H$

  m – Hermit rank

- Yes if $H^*(m) = \frac{1}{2}$,

$$\frac{1}{\sqrt{\varepsilon |\ln \varepsilon|}} \int_0^t G(Y_{s/\varepsilon})\, dB_s \to W_t$$

- If $H^*(m) > \frac{1}{2}$, $\varepsilon^{H^*(m)-1} \int_0^t G(Y_{s/\varepsilon})\, ds \to c Z_t^{H_*(m), m}$

Functional CLT   Gehringer – Li

formulation in rough path topolgy for any FCLT.

For $H > \frac{1}{2}$, this means weak convergence in C^α topolgy, $\alpha > \frac{1}{2}$.

We proved $dx_t^\varepsilon = \sum \varepsilon^i G_i(Y_t^\varepsilon) f_i(x_t^\varepsilon)$

Young integration

$f \in C^\alpha, \quad g \in C^\beta, \quad \alpha+\beta = 1$

$$\int_s^t f_r \, dg_r = \lim_{\substack{|P_n| \to 0}} \sum_{[u,v] \subset P} f_{u^*} g_{uv} \qquad u^* \in [u,v]$$

$$g_{uv} = g_v - g_u$$

$\int_s^t f_r \, df_r \qquad \alpha > \frac{1}{2}.$

$|(x,y)|_\alpha \sim n^{2\alpha - 1} \xrightarrow{n\to\infty} 0, \ \alpha < \frac{1}{2}$ in $C([0,T];R)$

$$\begin{cases} x_t^n = \frac{1}{n} \sin n^2 t & \longrightarrow 0 \\ y_t^n = \frac{1}{n} \cos n^2 t & \longrightarrow 0 \end{cases}$$

$$\int_0^t \left(\frac{1}{n} \sin(n^2 s)\right) dy_s^n = -\int_0^t (\sin(n^2 s))^2 \, ds$$

$$= -\int_0^t \frac{1}{2}(1 - \cos(2n^2 s)) \, ds \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha$$

$$= -\frac{t}{2} + \frac{\sin(2n^2 t)}{4} \not\to 0$$

$(x,y) \longmapsto \int_0^t x_s \, dy_s$ is not continuous from $C([0,T];R) \times C([0,T];R)$ to $C([0,T];R)$.

There does not exist a norm on smooth paths $\subseteq C([0,T];R)$ s.t. ① $\overline{C^\infty}$ contains all Brownian paths.

$\mu(\overline{C^\infty}) < 1$

Wiener measure

② the integral $C^\infty \times C^\infty$ extends ctsly to $\overline{C^\infty} \times \overline{C^\infty}$ with respect to this norm.

$\dot{x}_t = f(x_t) \, dY_t$

Enhance or lift a Hölder cts path $(X, \mathbb{X})$. Suppose $Y_s = F(x_s)$.

$$\int_s^t F(x_u) \, dx_u = F(x_s) x_{s,t} + \int_s^t \int_s^u (F(x_u) - F(x_s)) \, dx_u$$

$$F(x_u) - F(x_s) = \int_s^u \underbrace{\frac{d}{dr}(F(x_r))}_{} dr = \int_s^u (DF)(x_r) \, dx_r$$

$$= (DF)(x_s) x_{s,u} + \int_s^u ((DF)(x_r) - (DF)(x_s)) \, dx_r$$

$$\int_s^t F(x_u)\, dx_u \quad \cdots$$

(#)
$$= F(x_s)\, x_{s,t} + DF(x_s) \underbrace{\int_s^t x_{s,u}\, dx_u}$$

$$\sum_{i,j} \partial_i F^j(x_s) \underbrace{\int_s^t \int_s^u dx_s^i\, dx_u^j}$$

$$\int_s^t x_{su}^i\, dx_u^j$$

$$+ D^2 F(x_s) \int_s^r \int_s^u \int_s^v dx_r\, dx_v\, dx_u + \cdots$$

Enhancement $\quad \underline{X} : (s,t) \longmapsto (X^1_{s,t}, X^2_{s,t}, \cdots)$

$$X^1_{s,t} = X_t - X_s$$

$X_0 = 0$

$$\int_s^t F(X_r)\, d\underline{X}_r = \lim_{|P|\to 0} \sum F(x_u)\, X^1_{u,v} + DF(x_u)\, X^2_{u,v} + \cdots + D^{N-1} F(x_u)\, X^N_{u,v}$$

$(X_{s,t}, \int_s^t x_{s,u}\, dx_u)$ an enhancement

1st level. iterated integrals up to a certain order

Tensor notation $\quad e_i \otimes e_j, \quad \{e_1, \cdots, e_n\} \subset \mathbb{R}^d$

$$\iint_{s<u<v<t} dx_u^i \otimes dx_v^j$$

$$(x^0, \cdots, x^N)(Y^0, \cdots, Y^N) = (x^0 Y^0, x^0 Y^1 + x^1 Y^0, x^0 Y^2 + x^1 Y^1 + x^2 \ldots)$$

$V^{\otimes 0} = \mathbb{R}$ $\qquad X^{-1} = \sum_0^N (-1)^n (X-1)^{\otimes n}$

$X^0_{s,t} \equiv 1$ $\qquad (X \cdot Y)_n = \sum_{k=0} X_k \otimes Y_{n-k}$

$T_1^N(V) \quad X_\cdot = (X^0, X^1, \cdots X^N)$

$X_{s,t} = X_s^{-1} \otimes X_t$

Chen's $\quad \underline{X}_{s,u} = \underline{X}_{s,t} \otimes \underline{X}_{t,u} \qquad (*)$

e.g. $\alpha > \frac{1}{3}, \quad N = 3, \quad n = 1, 2, 3.$

$$\|X^n\|_{n\alpha} = \sup \frac{|X^n_{s,t}|}{(t-s)^{n\alpha}}$$

$$X^3_{s,u} = \int_{s<t_1<t_2<t_3<t} dx_{t_1} \otimes dx_{t_2} \otimes dx_3 = \underbrace{\int_{s<t_1<t_2} dx_{t_1} \otimes dx_{t_2}}_{X^2_{s,t}} \otimes \underbrace{\int_{t_2<t_3<t} dx_3}_{X^{n-2}_{t,u}}$$

Rough path metric

$X, \tilde{X}$ rough; $\rho_\alpha(X, \tilde{X}) = \sum_{n=1}^{N_\alpha} \|X^n - \tilde{X}^n\|_{n\alpha}$

- Geometry $X : \Delta_T \to V$ if $\exists\ x^{(m)}$ cts of bdd variation s.t. $\rho(x^{(m)}, X) \to 0$
  └ canonical lift

- Canonical lift of a smooth path X

$$\underline{X}_{s,t} = (1, X^1_{s,t}, \cdots, X^{N_\alpha}_{s,t})$$

$$X^n_{s,t} = \int_{s<t_1<\cdots<t_n<t} \dot{x}_{t_1} \otimes \cdots \otimes \dot{x}_{t_n}\, dt_1 \cdots dt_n$$

Here all $X^n_{s,t}$ are determined by first level path.

- weakly geometry if $\underline{X} \in G^N(V)$

$$G^N(V) = \{ \underset{(1, \xi_1, \cdots, \xi_N)}{\underline{\xi}} : \xi_m \times \xi_n = \sum_{\sigma \in S(m,n)} P_\sigma(\xi_{m+n}) \}$$

chen's identity $\quad \underline{X}_{s,u} \underline{X}_{u,t} = \underline{X}_{s,t}$

$$\|f\|_\alpha = \sup_{\substack{s\neq t\\ s,t\in I}} \frac{\|f(s)-f(t)\|}{|t-s|^\alpha} \ (+\|f\|_\infty), \quad I \text{ fixed}$$

C^{α}; If $x:[0,T]\to V$, define $x_{s,t}=x_t-x_s$ $0\le s\le t\le T$.

Def. Two parameter process: $\Delta=\{(s,t): 0\le s\le t\le T\}$

$A:\Delta\to V$, V Banach spaces, Hölder norm on I

$$\|A\|_\alpha = \sup_{\substack{u<v\\ u,v\in I}} \frac{\|A_{u,v}\|}{\|u-v\|^\alpha}$$

Denote: $C^\alpha(\Delta;V)=\{A:\Delta\to V \mid \|A\|_\alpha<\infty\}$.

If $s<u<t$, $\delta A_{s,u,t}=A_{s,t}-A_{s,u}-A_{u,t}$.

Def. $\alpha\in(\frac{1}{3},\frac{1}{2}]$. A rough path taking values in V of regularity α is a pair $\mathbf{X}\doteq(X,\mathbb{X})$ with $X\in C^\alpha([0,T];V)$ and $\mathbb{X}\in C^{2\alpha}(\Delta;V\otimes V)$ satisfying Chen's relation:

$\mathbb{X}^{i,j}_{s,t}=\mathbb{X}^{i,j}_{s,u}+\mathbb{X}^{i,j}_{u,t}+X^i_{s,u}\otimes X^j_{u,t}$

$$\delta\mathbb{X}_{s,u,t}:=\mathbb{X}_{s,t}-\mathbb{X}_{s,u}-\mathbb{X}_{u,t}=X_{s,u}\otimes X_{u,t}.$$

— $\mathscr{C}^\gamma([0,T];V)$ — $X_0=0$

$$\|\mathbf{X}\|_\alpha=\|X\|_\alpha+\|\mathbb{X}\|_{2\alpha}$$

<u>Controlled rough paths</u>

$X\in C^\alpha([0,T];V)$, $Y\in C^\alpha([0,T];W)$ is controlled by X if $\exists$ $Y'\in C^\alpha([0,T];\mathcal{L}(V,W))$ s.t. if

$$R^Y_{s,t}=Y_{s,t}-Y'_sX_{s,t},$$

$R\in C^{2\alpha}(\Delta;W)$, (Y,Y') a <u>controlled rough path</u> $\mathcal{D}_X^{2\alpha}([0,T];W)$.

By (#) on page 2.

$$Y_u = F(x_u), \qquad \overset{Y_t}{F(x_u)} = \overset{Y_s}{F(x_s)} \ \overset{Y_s' X_{s,u}}{DF(x_s) X'_{s,u} X'_{s,t}}$$

$$+ R_{s,u}$$

<u>Controlled</u> if R is regular $(N-i)\alpha$, $i = 0, \cdots, N-1$

$$\|Y\|_{X,\alpha} = \sum_{i=0}^{N-1} \|R^{Y,i}\|_{(N-i)\alpha} + \sum_{i=0}^{N-1} |Y_0^i|$$

semi-norm

$$d_{X,\bar{X}}(Y, \bar{Y}) = \sum_{i=0}^{N-1} \|R^{Y^i} - R^{\bar{Y}^i}\|_{(N-i)\alpha}$$

$$X \in D_X^{2\alpha}(Y), \quad \bar{X} \in D_X^{2\alpha}(\bar{Y})$$

$$F(Y_t^0) = F(Y_s^0) + DF(Y_s^0)(Y_{s,t}^0)$$

$$+ \int_0^1 (1-\theta) D^2 F(Y_s^0 + \theta Y_{s,t}^0)\underbrace{(Y_{s,t}^0)^{\otimes 2}}_{|t-s|^{2\alpha}} d\theta$$

$$Y \in D_{X,\alpha}(0), \quad Y_{s,t}^0 = Y_s' X_{s,t}' + R_{s,t}^Y$$

$$F(Y_t^0) = F(Y_s^0) + DF(Y_s^0) Y_s' X_{s,t}'$$

Stability after composition with a smooth function

$F(x_t) = F(x_s) - DF(x_s)x_s' X_{s,t}$

$F(Y_t^0)$ has derivative $DF(Y_t^0)Y_t'$

$d_{X,\tilde{X}}(F(Y), F(\tilde{Y}))$

$\rho_\alpha(X,\tilde{X}) + d_{X,\tilde{X}}(Y,\tilde{Y})$

$\|Y_0^0 - \tilde{Y}_0^0\| + \|Y_0' - \tilde{Y}_0'\|$

$I_t = \int Y\,dx \cdots$ $\quad I_t^0 = \int_0^t Y_s\,dx_s, \quad I_t' = Y_t^0$

$\alpha \in (\frac{1}{3}, \frac{1}{2}]$ $\quad \int Y\,dx = \lim \sum_{[u,v]\in\mathcal{P}} (Y_u^0 X_{u,v} + Y_u' \mathbb{X}_{u,v}^2)$

Continuity of rough integrals (controlled integrands)

$d(Y,\tilde{Y}) + \rho_\alpha(X,\tilde{X}) + |Y_0 - \tilde{Y}_0|$

Rough integral $= Y_t = Y_0 + \int_0^t F(Y_s)\,dx_s$

fixed point theorem.

$Y = (Y^0, Y') \in \mathcal{D}_X^\alpha(U)$ is a solution, if

$Y_t^0 = Y_0 + (\int_0^\cdot F(Y)\,dx)_t^0$

$Y_t^i = (\int_0^\cdot F(Y)\,dx)_t^i$ $\quad 1 \le i \le N-1$

$Y_t' = F(Y_t^0)$. If Y solves RDE its high level components are determined by Y_t^0

Cty of solutions

cty of solutions of RDE. $(x_0, f, \mathbf{x})$ $dX_t = f(X_t)\, d\mathbf{X}_t$ (7)

Gaussian rough path $\sim$ fBM, $H \in (\frac{1}{3}, \frac{1}{2}]$.

$$dY_t = f(Y_t)\, dB_t^H + g(Y_t)\, dt$$

Piecewise linear $\mathbb{X}_{s,t} = \int_s^t \tilde{X}_{s,u}\, d\tilde{X}_{s,t}$ $\tilde{X}_t =$ piecewise constant

This defines a a.s. rough path lift

Cauchy in rough path topology

$\rho_\alpha(\mathbf{X}, \mathbf{Y}) = \|X - Y\|_\alpha + \|\mathbb{X} - \mathbb{Y}\|_{2\alpha}$

$\frac{T}{2^m}$ $\frac{2T}{2^m}$

For BM, this coincides with $\int_s^t W_{s,r}^i \circ dW_r^j$ stratonovich integral

$$dY_t = \underset{\text{RDE}}{f(Y_t)\, d\mathbf{W}_t} = f(Y_t)\, dW_t + \frac{1}{2}\sum_j \frac{\partial f_k}{\partial y_j} f_k^j(Y_t)\, dt$$

$$\frac{1}{2} Df_k(f_k)\, dt$$

$\mathbf{X} = (X, \mathbb{X})$, $X_{s,t} = X_t - X_s$

BM: $\mathbb{X}_{s,t} = \int_s^t X_{s,r}\, dX_r$ Itô / stratonovich $\to$ geometric

$(1, X, \mathbb{X})$ rough $\gamma \in (\frac{1}{3}, \frac{1}{2})$

fBM, covariance f satisfies a finite $\frac{1}{2H}$-variation condition

$\to$ Canonical p-geometric rough path

Lemma $H \in (\frac{1}{3}, \frac{1}{2})$, $f \in C^\beta$, $\beta > (1-2H) \vee 0$

$Z_{s,t}^f = \int_s^t f(r)\, dB(r)$ has a geometric RP lift

if $\mathbb{E}(|Z_{s,t}^f|^2) \le |f|_\infty |f|_{C^\beta} |t-s|^{2H}$

$\mathbb{E}\left(Z_{s,s+h}^f(x)\, Z_{t,t+h}^f(x)\right| \le \|f\|_\infty^2 |t-s|^{2H-2} h^2$.

$h \le t-s$.

Sewing lemma Suppose $(\Xi_{s,t})$

$|\Xi_{s,t}| \le K|t-s|^\eta$, $|\delta \Xi_{s,u,t}| \le K|t-s|^{\bar{\eta}}$, $\eta > 0$, $\bar{\eta} > 1$

$I_{s,t}(\Xi) := \lim_{|P| \to 0} \sum \Xi_{u,v}$ exists.

$|I_{s,t}(\Xi) - \Xi_{s,t}| \le \bar{K} |t-s|^{\bar{\eta}}$.

Stochastic sewing lemma

$$\|A\|_{\eta,p} = \sup_{s<t} \frac{\|A_{s,t}\|_p}{|t-s|^\eta} < \infty.$$

$$|||A|||_{\eta,p} = \sup_{s<u<t} \frac{\|\mathbb{E}(\delta A_{sut} | \mathcal{F}_s)\|_p}{|t-s|^\eta} < \infty$$

$\eta > \frac{1}{2}$, $\hat{\eta} > 1$. Then $I_{s,t}(A) = \lim_{|P|\to 0} \sum A_{u,v}$

exists. $\quad I_{s,u}(A) + I_{u,t}(A) = I_{s,t}(A)$

If $\|\mathbb{E}(A_{s,t} | \mathcal{F}_s)\|_p \leq |t-s|^{\hat{\eta}} \Rightarrow I(A) = 0$

$$\|I_{s,t}(A)\|_p \leq C\left(|||A|||\, |t-s|^{\hat{\eta}} + \|A\|_{\eta,p}\, |t-s|^\eta\right).$$

Langevin Equation in 1-dimension

$$\begin{cases} \dot{x}_t = v \\ \varepsilon \dot{v} = -\lambda v \end{cases}$$

small mass

Example
$$\begin{cases} \dot{x}_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} Y_t^\varepsilon \\ dY_t^\varepsilon = -\frac{\lambda}{\varepsilon} Y_t^\varepsilon dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t \end{cases}$$

$$x_t^\varepsilon = x_0 + \frac{1}{\sqrt{\varepsilon}}\int_0^t Y_r^\varepsilon dr = \frac{1}{\lambda}\varepsilon(Y_0 - Y_t^\varepsilon) + \underbrace{(\sigma W_t + x_0)}\frac{1}{\lambda}$$

Fractional Langevin equation.

$$\begin{cases} \dot{x}_t^\varepsilon = \varepsilon^{H-1} y_t^\varepsilon, \quad x_0^\varepsilon = x_0 \\ dy_t^\varepsilon = -\frac{\lambda}{\varepsilon} y_t^\varepsilon dt + \frac{\sigma}{\varepsilon^H} dB_t^H \\ Y_0^\varepsilon = \varepsilon^{-H}\sigma\int_{-\infty}^0 e^{\lambda\frac{s}{\varepsilon}} dB_s^H = \varsigma \end{cases}$$

$$\begin{aligned} &\dot{x}_t = v \\ &\varepsilon dv_t = \lambda v_t dt + \sigma dB_t^H \\ &\lambda v_t dt = \sigma dB_t \ (\varepsilon = 0) \\ &x_t = \frac{\sigma}{\lambda} B_t^H \end{aligned}$$

$$|y_t^\varepsilon - y_t^{\varepsilon,\varsigma}| = e^{-\frac{\lambda t}{\varepsilon}} |y_0 - \varsigma| \xrightarrow[(\varepsilon\to 0)]{} 0, \quad y_t^\varepsilon \text{ a solution}$$

In general, convergence rate of two solutions do not translate to rate to equilibrium

$$y_t^\varepsilon = y_{t/\varepsilon} \qquad dY_t = -\lambda Y_t dt + \sigma dB_t^H$$

Lemma $\gamma \in (0,H)$, $p \geq 2$, $Y_0 \in L^p$. $\mathbb{E}|Y_s - Y_t|^p \leq (2p-1)^{p/2}(1+\lambda+\sigma)\left(2\sup_r \|Y_r\|_{L^2}\right)|t-s|^{2pH}$.

Prop. $H \in (0,1)\setminus\{\frac{1}{2}\}$, $\gamma \in (0,H)$, $p \geq 2$, $T > 0$.

dim=1

$$y_0 \in L^p. \quad X_t^\varepsilon = \varepsilon^{H-1}\int_0^t Y_r^\varepsilon dr$$

Then
$$\sup_{s,t} \|X_{s,t}^\varepsilon - \frac{\sigma}{\lambda} B_{s,t}^H\|_{L^p} \leq \varepsilon^H$$

$$\left\| \|X_t^\varepsilon - \frac{\sigma}{\lambda} B_t^H\|_{C^{\gamma'}([0,T];\mathbb{R})} \right\|_{L^p} \lesssim \varepsilon^{H-\gamma'}$$

for $\forall \gamma' < \gamma$. So $X_t^\varepsilon \to \frac{\sigma}{\lambda} B_t^H$ weakly in $C^{\gamma'}$.

Proof
$$X_{s,t}^\varepsilon = X_t^\varepsilon - X_s^\varepsilon = \varepsilon^{H-1}\int_s^t Y_r^\varepsilon dr$$
$$= \frac{\varepsilon^H}{\lambda}(-y_t^\varepsilon + y_s^\varepsilon) + \frac{\sigma}{\lambda} B_{s,t}^H$$

$$\sup_{s,t\in[0,T]} \|y_t^\varepsilon - y_s^\varepsilon\|_{L^p} \leq \sup_{s,t\in[0,\frac{T}{\varepsilon}]} \|y_t - y_s\| < \infty$$

$$\frac{\varepsilon^H}{\lambda}(y_t^\varepsilon - y_s^\varepsilon) \to 0 \quad \text{in } L^p(\Omega).$$

$$\| X_{s,t}^\varepsilon - \frac{\sigma}{\lambda} B_{s,t}^H \|_{L^p(\Omega)}$$

$$= \frac{\varepsilon^H}{\lambda} \| y_t^\varepsilon - y_s^\varepsilon \|_{L^p} = \frac{\varepsilon^H}{\lambda} \| y_{t/\varepsilon} - y_{s/\varepsilon} \|_{L^p}$$

$$\lesssim \varepsilon^{H-\lambda} |t-s|^{2\gamma}$$

By Kolmogorov's Thm

$\forall s,t$, $\mathbb{E}\| \Theta(t) - \Theta(s) \|^p \le C_p |t-s|^{p\beta}$, $C_p > 0$,

Then $\| |\Theta|_\gamma \|_{L^p(\Omega)} \lesssim C_p \sum_{n \in \mathbb{N}_0} \frac{1}{2^{n(\beta - \frac{1}{p} - \gamma)}}$

$< \infty$.

$$\Rightarrow \| \| X^\varepsilon - \frac{\sigma}{\lambda} B^H \|_{C^{\gamma'}([0,T];\mathbb{R})} \|_{L^p(\Omega)} \le \varepsilon^{H-\gamma}.$$

$$* \; dx_t^\varepsilon = f(x_t^\varepsilon)\, \varepsilon^{H-1}\, dy_t^\varepsilon$$

Def. ① Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. A rough path taking values in V of regularity α is a pair $\underline{X} = (X, \mathbb{X})$ with $X \in C^\alpha([0,T]; V)$ and $\mathbb{X} \in C^{2\alpha}(\Delta; V \otimes V)$,

$$\|\mathbb{X}\|_{2\alpha} = \sup_{u<v} \frac{\|\mathbb{X}_{u,v}\|}{\|u-v\|^{2\alpha}}$$

Satisfying Chen's relation.

$$\|\underline{X}\|_\alpha = \|X\|_\alpha + \|\mathbb{X}\|_{2\alpha}.$$

$\int_s^t Y_r \, dX_r$ $\qquad W = \mathcal{L}(V, U)$

② $\underline{Y}_t = (Y_t, Y_t')$, where $Y \in C^\alpha([0,T]; W)$ and $Y' \in C^\alpha([0,T]; \mathcal{L}(V,W))$, is a α-Hölder path controlled by X if

$$Y_{s,t} = Y_s' X_{s,t} + R^Y_{s,t}$$

and $R \in C^{2\alpha}(\Delta; W)$ $\qquad D_X^{2\alpha}([0,T]; W)$

(Recall $\underline{F(X_t) - F(X_s)} = (DF)_{X_s}(X_{s,t}) + \square$)

$\mathcal{L}(V, \mathcal{L}(V,U)) \sim V \otimes V$

Rough integral $\qquad \int_s^t F(X_r)\,dX_r = F(X_s) X_{st} + DF|_{X_s}\left(\int_s^t X_{sr}\,dX_r\right) + \cdots$

$$\int_s^t Y_r \, dX_r = \lim_{|P| \to 0} \sum_{[u,v] \in P} Y_u X_{uv} + Y_u' \mathbb{X}_{u,v} \quad \text{exists in } U.$$

$\underline{I}_t(Y, X) = \left(\int_0^t Y_r \, dX_r, Y_t \right)$ is a U-valued α-rough path controlled by X.

$$\|I\|_\alpha \lesssim T^\alpha (1 + \|\underline{X}\|_\alpha)(\|\underline{Y}\|_\alpha + |Y_0'| \cdot \|X\|_\alpha)$$

$$d\left(\int_0^t Y_r\,dx_r\ ,\ \int_0^t \tilde{Y}_r\,d\tilde{x}_r\right)$$

$$\lesssim \left(d(Y,\tilde{Y}) + \rho_\alpha(x,\tilde{x}) + |Y_0' - \tilde{Y}_0'|\right)$$

C depend on bdds on norms.

RDE $\quad Y_t = F(Y_t)\,d\mathbf{X}_t\ ,\quad Y_0$

Thm. If $F \in C_b^3(W, L(U,W))$. Then there exists a controlled rough path of regularity α (by $\mathbf{X}$)

$$Y_t = Y_0 + \int_0^t F(Y_s)\,dx_s. \qquad \text{Set}$$

$$Y_t' = \left(\int_0^t F(Y_s)\,dx_s\right)' = F(Y_t)$$

s.t. $\Phi(Y_0, \mathbf{X}) : W \times \mathscr{C}^\gamma([0,T];U) \longrightarrow C^\gamma([0,T],W)$

Lipschitz cts, Lipschitz constant depends on $f, \gamma, \|\mathbf{X}\|_\gamma, \|\mathbf{X}'\|_\gamma$.

The "derivative" of the solution path are ! determined by its zeroth level.

~~Thm For each $Y_0 \in U$, $\exists !\ Y \in D_X^{2\alpha}(U)$~~

$$\begin{cases} dY_t = F(Y_t)\,dx_t\ , & Y_0 \\ d\tilde{Y}_t = F(\tilde{Y}_t)\,d\tilde{x}_t & \tilde{Y}_0 \end{cases}$$

$$d(Y,\tilde{Y}) \leq C\left(\rho_\alpha(\mathbf{x},\tilde{\mathbf{x}}) + |Y_0 - \tilde{Y}_0|\right).$$

Corollary $\quad f \in C_b^3,\ \mathbf{X}^\varepsilon \in \mathscr{C}^\gamma([0,T],\mathcal{X}) \xrightarrow{(W)} \mathbf{X} \in \mathscr{C}^\gamma([0,T];\mathcal{X})$.

$dx_t^\varepsilon = f(x_t^\varepsilon)\,d\mathbf{X}_t^\varepsilon,\ x_0^\varepsilon = x_0$. As $\varepsilon \to 0$, if $\mathbf{X}^\varepsilon \xrightarrow{(W)} \mathbf{X}$,

then $x_t^\varepsilon \xrightarrow[\text{weakly}]{} x_t$ in C^γ, $\quad dx_t = f(x_t)\,d\mathbf{X}_t$, same initial condition

x_t solves

Enhancing a rough path,

Canonical lift of a smooth path

$$\gamma \longmapsto \left(1, \gamma_{s,t}, \int_s^t \gamma_{s,u}\, d\gamma_u, \int_s^t\int_s^{u_1}\int_s^{u_2} \dot\gamma_{u_3}\dot\gamma_{u_2}\dot\gamma_{u_1} \dots\right)$$

$$\hat\gamma_{s,t} = \int_{s<t_1<\dots<t_n<t} \dot x_{t_1}\otimes\dots\otimes \dot x_{t_n}\, dt_1 \dots dt_n$$ completely determined.

Stochastic processes.

$W_t \longrightarrow$ There $\exists$ a set of full measure, $\frac{1}{3}<\alpha<\frac{1}{2}$

$W_t(\omega)$ has a roughpath life.

$$\left(W_{s,t}(\omega), \int_s^t W_{s,u}\, dW_u\right)$$

$$\parallel$$

Itô $\quad \int_s^t (W_u - W_s)\, dW_u = \frac{1}{2}\left(W_t^2 - W_s^2 - (t-s)\right) - W_s(W_t - W_s)$

Stratonisch $\quad \int_s^t (W_u - W_s)\circ dW_u = -\frac{1}{2}(W_t^2 - W_s^2) - W_s(W_t - W_s)$

$$= -(W_t - W_s)\left(\tfrac{1}{2}W_t + \tfrac{1}{2}W_s - W_s^2\right)$$

for a full set of ω, W is C^α, its lift $\in C^{2\alpha}(\Delta, V\otimes V)$

Gaussian processes (independent component) has a lift.

Gaussian processes with ~~origin~~ independent components.

geometric lifts : $\quad Sym(\mathbb{X}_{s,t}) = \frac{1}{2} X_{s,t}\otimes X_{u,t}$

$$Sym(\mathbb{X}_{s,t}) = \frac{1}{2}(\mathbb{X}_{s,t} + \mathbb{X}_{t,s})$$

$$\mathbb{X}^{i,j}_{s,t} = \begin{cases} \int_s^t X^i_{s,r}\, dX^j_r & i\neq j \\ \frac{1}{2}(X^i_{s,t})^2 & i=j \end{cases}$$

define an integral in L^2 with $\lim\limits_{|P|\downarrow 0} \sum_{[u,v]\subset P}' X^i_u X^j_{uv}$

Then find a modification

$R^{i,j} = \mathbb{E}(X^i_{s,t} X^j_{s',t'})$ $\qquad \left|R\binom{s,t}{s,t}\right| \leq M|t-s|^{\frac{1}{\rho}} \Rightarrow X_{s,t}\in C^{\frac{1}{2\rho}-}$

$\mathbb{X}_{s,t}$ defined in L^2, $\alpha<\frac{1}{2\rho}$, get geometric α-Rough path. $\frac{1}{2\rho} = H$

Let $dy_t^\varepsilon = -\frac{\lambda}{\varepsilon} y_t^\varepsilon dt + \frac{\sigma}{\varepsilon^H} dB_t^H$ $\qquad y_t^\varepsilon = y^1_{t/\varepsilon}$

$$y_0^\varepsilon = \varepsilon^{-H} \sigma \int_{-\infty}^0 e^{\frac{\lambda s}{\varepsilon}} dB_s^H \qquad = \varepsilon^{-H} \sigma \int_{-\infty}^0 e^{\frac{\lambda s}{\varepsilon}} dD(W_s)$$

$$y_t^\varepsilon = e^{-\frac{\lambda t}{\varepsilon}} \varepsilon^{-H} \sigma \int_{-\infty}^0 e^{\frac{\lambda s}{\varepsilon}} dB_s + \sigma/\varepsilon^H \int_0^t e^{-\frac{\lambda(t-s)}{\varepsilon}} dB_s^H$$

$$= \varepsilon^{-H} \sigma \int_{-\infty}^t e^{-\frac{\lambda(t-s)}{\varepsilon}} dB_s^H \quad \sim y_0^\varepsilon$$

Consider $dx_t^\varepsilon = \sum_{k=1}^N \alpha_k(\varepsilon) f_k(x_t^\varepsilon) G_k(y_t^\varepsilon) dt$

$f = (f_1, \cdots f_N)$, $\qquad dx_t^\varepsilon = f(x_t^\varepsilon) dX_t^\varepsilon$

$$\boxed{\begin{aligned} &X_{s,t}^\varepsilon = \left(\alpha_1 \int_s^t G_1(y_r^\varepsilon) dr, \; \cdots, \; \alpha_N \int_s^t G_N(y_r^\varepsilon) dr\right) \\ &\mathbb{X}_{s,t}^{i,j,\varepsilon} = \int_s^t X_{s,r}^{i,\varepsilon} \, dX_r^{j,\varepsilon}. \end{aligned}}$$

$\bar H(k) = m_k(H-1) + 1$, $\qquad m_k$ = Hermite rank of G_k.

$$G_k = \sum_{j=m_k}^\infty c_k^j H_j$$

Order G so that $1 \le m_1 \le m_2 \le \cdots$.

$n^* = \inf_k \{ k(H-1) + 1 \le \frac{1}{2} \}$.

$$\underbrace{G_1 \; - - \; G_{n-1}}_{n-1 < n^*}, \; \underbrace{G_n, \; \cdots, \; G_N}_{n \ge n^*} \qquad Y_s$$

$\bar H(m_k) \le \frac{1}{2}$ $\quad$ Hölder $> \frac{1}{2}$ $\quad$ Hölder $< \frac{1}{2}$ $\quad$ $\bar H(m_k) > \frac{1}{2}$

$\alpha_k(\varepsilon) = \varepsilon^{\bar H(m_k) - 1}$ $\qquad \varepsilon^{-1/2}$

If $H \le \frac{1}{2}$, $k=1$, $\quad H - 1 + 1 \le \frac{1}{2}$.

Theorem 1 Suppose that $H \neq \frac{1}{2}$, G_k has Hermite rank ≥ 1

Set
$$\alpha_k(\varepsilon) = \begin{cases} \varepsilon^{-\frac{1}{2}} & \bar{H}(m_k) < \frac{1}{2} \\ \varepsilon^{-\frac{1}{2}} \frac{1}{\sqrt{|\ln|\varepsilon||}}, & \bar{H}(m_k) = \frac{1}{2} \\ \varepsilon^{\bar{H}(m_k) - 1}, & \bar{H}(m_k) > \frac{1}{2} \end{cases}$$

$\bar{H}$ small $\Leftrightarrow$ Hermit rank high

Then $X_t^\varepsilon \to (\Sigma W_t, Z_t)$ in finite dimensional distributions.

$$\Sigma^2_{i,j} = \begin{cases} \sum_q 2 c_{i,q} c_{j,q}\, q! \int_0^\infty \rho^q(s)\, ds, & \bar{H}(m_i) \vee \bar{H}(m_j) < \frac{1}{2} \\ 0, & \bar{H}(m_i) = \frac{1}{2} \text{ or } \bar{H}(m_j) = \frac{1}{2}, \quad \bar{H}(m_i) \wedge \bar{H}(m_j) < \frac{1}{2} \\ 2 c_{j,m_j} c_{i,m_i}\, m_j! \left(\frac{H(2H-1)\sigma^2}{\lambda^2} \right)^{m_j} & \text{otherwise} \end{cases}$$

Z_t is independent of W_t

Z_t is Hermit process given by $c_k Z_t^{\bar{H}(m_k), m_k}$ $\quad k = n+1, \dots, N$.

with same BM generating it

If G_k is nice, e.g. $G_k \in L^{p_k}$, $p_k > 2$.

$X_t^\varepsilon \xrightarrow{W}$ in C^γ, $\quad \gamma < \min\limits_{1 \le k \le n} \left(\frac{1}{2} - \frac{1}{p_k}\right) \wedge \min\limits_{k = n+1, \dots, N} \left(\bar{H}(m_k) - \frac{1}{p_k}\right)$

$\bar{H}(m_k) > \frac{1}{2}$. $\quad$ by definition.

Thm 2. G_k satisfies fast chaos decay condition of rate λ p_k

$G = \sum\limits_{\ell \in \mathbb{N}_0^m} c_\ell H_\ell \qquad \sum\limits_{\ell \in \mathbb{N}_0^m} |c_\ell| \sqrt{\ell!}\, (p-1)^{\frac{|\ell|}{2}} < \infty$

$\bar{H}(m_k) < 0 \qquad \bar{H}(m_{n+1}) \geq \frac{1}{2}$

$X^\varepsilon \to C^\gamma$ where $\gamma > \frac{1}{3}$

Thm 3 Under condition of Thm 2.

$$dx_t^\varepsilon = \sum_{k=1}^{N} f(x_t^\varepsilon)\, \alpha_k(\varepsilon)\, G_k(Y_t^\varepsilon)\, dt$$

$x_t^\varepsilon \xrightarrow{C^r} x_t$, x_t solves $dx_t = f(x_t)\, d\underline{X}_t$

$$\Leftrightarrow dx_t = \sum_{k=1}^{n} f_k(x_t) \circ d\left(\Sigma W_t\right)_k + \sum_{k=n+1}^{N} f_k(x_t)\, d z_t^k$$

Remark $\quad dx_t = \alpha_{..}(\varepsilon)\, f(x_k^\varepsilon)\, G_{..}(Y_t^\varepsilon)\, dt$

$H(m) \le \frac{1}{2}$, $\quad x_0^\varepsilon = x_0 \in R^d$

Sufficient $f \in \mathrm{Lip}$, G is cts $G \in L^{2+}$.

Lemma 2 Suppose $\gamma \in (\frac{1}{3}, 1)$, $f \in C_b^3(\mathbb{R}^d, \mathbb{R}^d)$, $G \in C^{0+}(\mathbb{R}^N, \mathbb{R}^d)$

$$dx_t = f(x_t) G(y_t) dt, \qquad y_t \in C^\gamma$$

Set $X_t = \int_0^t G(y_s) ds$

$$\mathbb{X}_{s,t} = \int_s^t X_{s,r} \otimes dX_r$$

Then x_t solves the RDE $dx_t = f(x_t) d\mathbf{X}_t$.

Pf

Lemma 1 (Sewing lemma) $T > 0$, $\gamma \in (0,1)$, $\bar{\eta} > 1$.

Suppose $\left\{ A : \Delta \to V \text{ is s.t. } |A|_{\gamma,\eta} = \sup_{0 \le s < t \le T} \frac{\|A_{s,t}\|}{|t-s|^\gamma} + \sup \frac{\|\delta A_{s,u,t}\|}{|t-s|^{\eta}} < \infty \right\} = C_2^{\gamma,\eta}$.

Then there exists a ! cts linear mapping $I : C_2^{\gamma,\eta} \to C^\gamma$

s.t. $I(A)_0 = 0$,

$$|I(A)_{s,t} - A_{s,t}| \le C|t-s|^\eta, \quad 0 \le s \le t \le T.$$

$$C(\bar{\eta}, |\Xi|_{\gamma,\eta}).$$

Proof Set $A_{s,t} = f(x_s) G(y_s)(t-s)$

$$\tilde{A}_{s,t} = f(x_s) X_{s,t} + f(x_s) Df(x_s) \mathbb{X}_{s,t}$$

Suppose $|A_{s,t} - \tilde{A}_{s,t}| \le C|t-s|^\eta$, $\eta > 1$.

Then $I(A) = I(\tilde{A})$

$$|A_{s,t} - \tilde{A}_{s,t}| = \left| f(x_s)\left(G(y_s)(t-s)\right) - f(x_s) X_{s,t} - f(x_s) Df(x_s) \mathbb{X}_{s,t} \right|$$

$$\lesssim \|f\|_\infty \left| \int_s^t (G(y_s) - G(y_r)) dr \right| + |Df \cdot f|_\infty \int_s^t \int_s^r G(y_p) G(y_r) \, dp \, dr$$

$$\lesssim \|f\|_\infty \|G\|_\delta \int_s^t |s-r|^{\delta\gamma} dr + \|Df\, f\|_\infty \|G\|_\infty^2 \int_s^t |r-s| dr$$

$$\lesssim |t-s|^{1+\delta\gamma}$$

Lemma Let $\theta \in (0,1)$. Suppose for p with $\theta - \frac{1}{p} > 0$

$$\|X^\varepsilon_{s,t}\|_{L^p} \lesssim |t-s|^\theta, \quad \|\mathbb{X}^\varepsilon_{s,t}\|_{L^{p/2}} \lesssim |t-s|^{2\theta} \quad \varepsilon \in (0,\tfrac{1}{2}]$$

Then $\sup_{\varepsilon \in (0,\frac{1}{2}]} \mathbb{E}\left[\|\underline{X}^\varepsilon\|_\gamma^p\right] < \infty$

for $\gamma < \theta - \frac{1}{p}$.

Lemma $\mathcal{C}^\gamma \subset \mathcal{C}^{\gamma'}$ is compactly embedded for $\frac{1}{3} < \gamma' < \gamma < 1$.

Lemma $\gamma \in (\frac{1}{3}, \frac{1}{2})$. Suppose $\sup_{\varepsilon \in (0,\frac{1}{2}]} \mathbb{E}\left(\|\underline{X}^\varepsilon\|_\gamma^p\right) < \infty$ some $p > 1$

Then: if the finite dimensional distributions of each weak limit of $\underline{X}^\varepsilon$ coincide, $\exists\, \underline{X} \in \mathcal{C}^{\gamma-}$ s.t. $\underline{X}^\varepsilon \to \underline{X}$ in $\mathcal{C}^{\gamma-}$

Proof. Show $\underline{X}^\varepsilon$ is tight, and finite dimensional distributions uniquely determine limit points.

Let $\underline{X}^1, \underline{X}^2$ be limit points.
Portmanteau theorem shows they are Radon.
Let $\mathcal{F} = \{ \underline{X} \mapsto \exp\left(i \sum_{j=1}^n \varphi_j(X_{t_j}, \mathbb{X}_{t_j})\right)$

$n,\ t_j,\ \varphi_j \in \mathbb{L}(U \oplus (U \otimes U), \mathbb{R}) \}$

- $\mathcal{F}$ separate points in $\mathcal{C}^{\gamma'}$
- closed under pointwise multiplication
- $1 \in \mathcal{F}$.
- $\mathbb{E}[f(\underline{X}^1)] = \mathbb{E}[f(\underline{X}^2)]$, $\forall f \in \mathcal{F}$.

$\Rightarrow \underline{X}^1 \overset{\text{law}}{=} \underline{X}^2$.

Tightness $K_M = \{\underline{X} : \|\underline{X}\|_\gamma \le M\}$ (bdd set), $\mathbb{P}(\|\underline{X}^\varepsilon\|_\gamma > M) \le \dfrac{\mathbb{E}\|\underline{X}^\varepsilon\|_\gamma^p}{M^p}$.

$\forall \delta > 0, \exists M$ s.t. $\mathbb{P}(\underline{X}^\varepsilon \notin K_M) < \delta$, K_M is relatively compact

Hairer

Thm 1 Consider $dx_t^\varepsilon = f(x_t^\varepsilon, Y_t^\varepsilon)\, dB_t^H + g(x_t^\varepsilon, Y_t^\varepsilon)\, dt$, $H > \frac{1}{2}$

Suppose $Y_t^\varepsilon = Y_{t/\varepsilon}$, $Y_t \perp\!\!\!\perp B$, strongly mixing rate $t^{-\delta}$

$$\sup\{\mathbb{P}(A \cap \bar{A}) - P(A)P(\bar{A}),\ A \in \sigma(Y_0), \bar{A} \in \sigma(Y_t)\} \lesssim t^{-\delta},\ \delta > 0.$$

Then $x_t^\varepsilon \to x_t$ in prob.

$$d\bar{x}_t = f(\bar{x}_t)\, dB_t + \bar{g}(x_t)\, dt.$$

Thm 2 $H \in (\frac{1}{3}, 1)$. $F^{(x,y)}$ supernice, derivative decays $C^4(x \mapsto F(x,\cdot))$ Lip.

$$dx_t^\varepsilon = \varepsilon^{\frac{1}{2}-H}\, F(x^\varepsilon, Y^\varepsilon)\, dB + F_0(x^\varepsilon, Y^\varepsilon)\, dt$$

makes sense

Y_t strongly cts semi-group, spectral gap in $\mathrm{Lip}(Y)$.

If $H > \frac{1}{2}$, assume $\int F_i(x,y)\, \mu(dy) = 0$, $\forall i \neq 0$.

$x_t^\varepsilon \to x_t$ Law.

$$dx_t = W(x_t, dt) + G(x_t)\, dt + \bar{F}_0(x_t)\, dt.$$

$$E\big(W_i(x,t), W_j(\bar{x}, \bar{t})\big) = (t \wedge \bar{t})\big(\Sigma_{i,j}(x, \bar{x}) + \Sigma_{j,i}(\bar{x}, x)\big)$$

$$G_i(x) = \frac{\partial}{\partial y}\Big|_{y=x} \Sigma_{ji}(x,y).$$

$$\Sigma_{ij} = \Gamma(2H+1)\ \langle \mathcal{L}^{1-2H} f_i, f_j \rangle_{L^2(\mu)}.$$

$$\mathcal{L}^{\alpha} f = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty t^{2H-2} P_t f\, dt \qquad H < \tfrac{1}{2}$$

$$\frac{1}{\Gamma(1-\alpha)} \int_0^\infty t^{2H-2} (P_t f - f)\, dt \qquad H > \tfrac{1}{2}$$

($\alpha = 1-2H$)