Aalto University
School of Science

Concentration of large random Gram matrices (ongoing work)

Lammi virtual poster presentation slides

Ian Välimaa, 25.–29.5.2026

Content

- Numerical studies
- Theorem
- Proof outline

Numerical studies

Simulate an $n \times m$-random matrix $\mathbf{X}$ with i.i.d. entries $X_{ij} \sim \mathrm{Ber}(p) - p$. Then

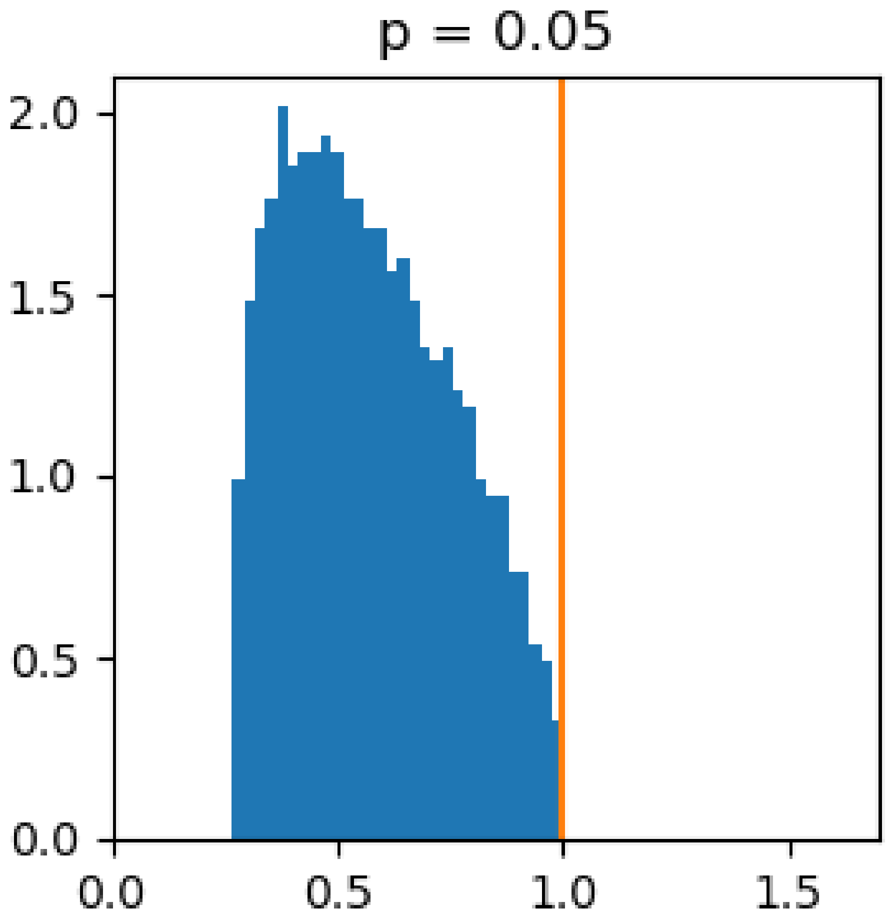
- $|X_{ij}| \leq 1$
- $\mathbb{E}X_{ij} = 0$
- $\mathbb{E}X_{ij}^2 = \sigma^2 = p(1-p)$

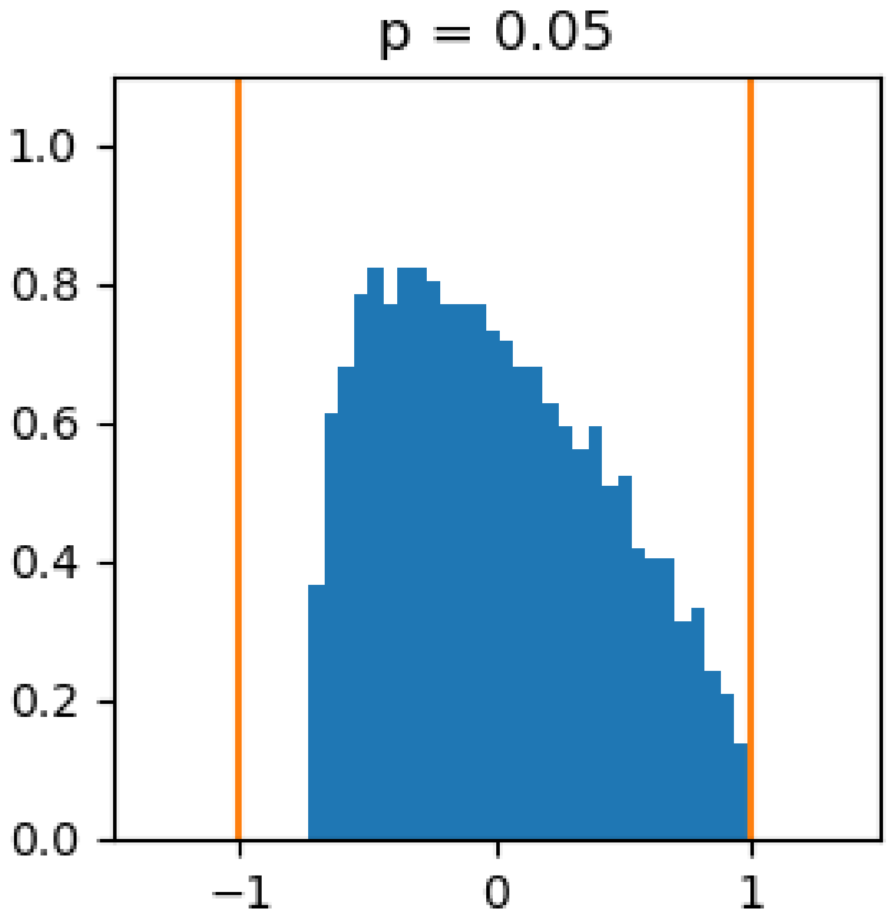
Compare the eigenvalues of $\mathbf{X}\mathbf{X}^\top$ and $\mathbf{X}\mathbf{X}^\top \odot \mathbf{M}$, where $\mathbf{M}$ removes the diagonal ($M_{ij} = 1$ if $i \neq j$ and $M_{ii} = 0$)

In the following, $n = 1000$ and $m = 10\,000$

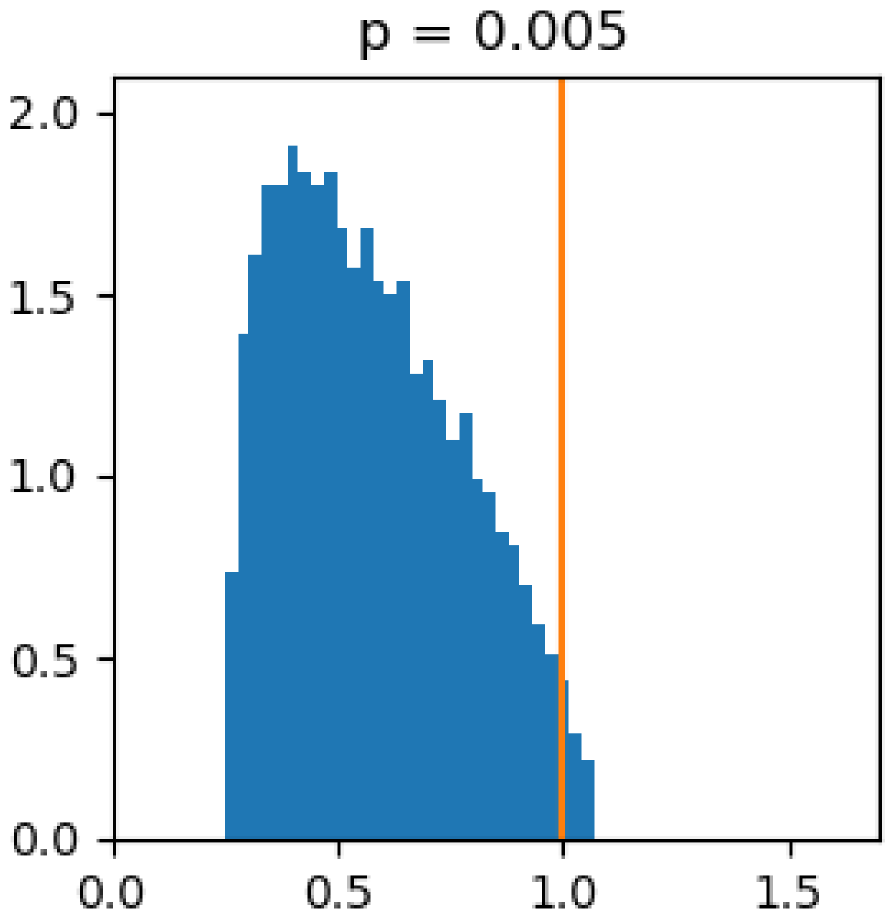
$$\frac{\mathbf{X}\mathbf{X}^\top}{(2\sqrt{nm} + n + m)\sigma^2}$$

$$\frac{\mathbf{X}\mathbf{X}^\top \odot \mathbf{M}}{(2\sqrt{nm} + n)\sigma^2}$$

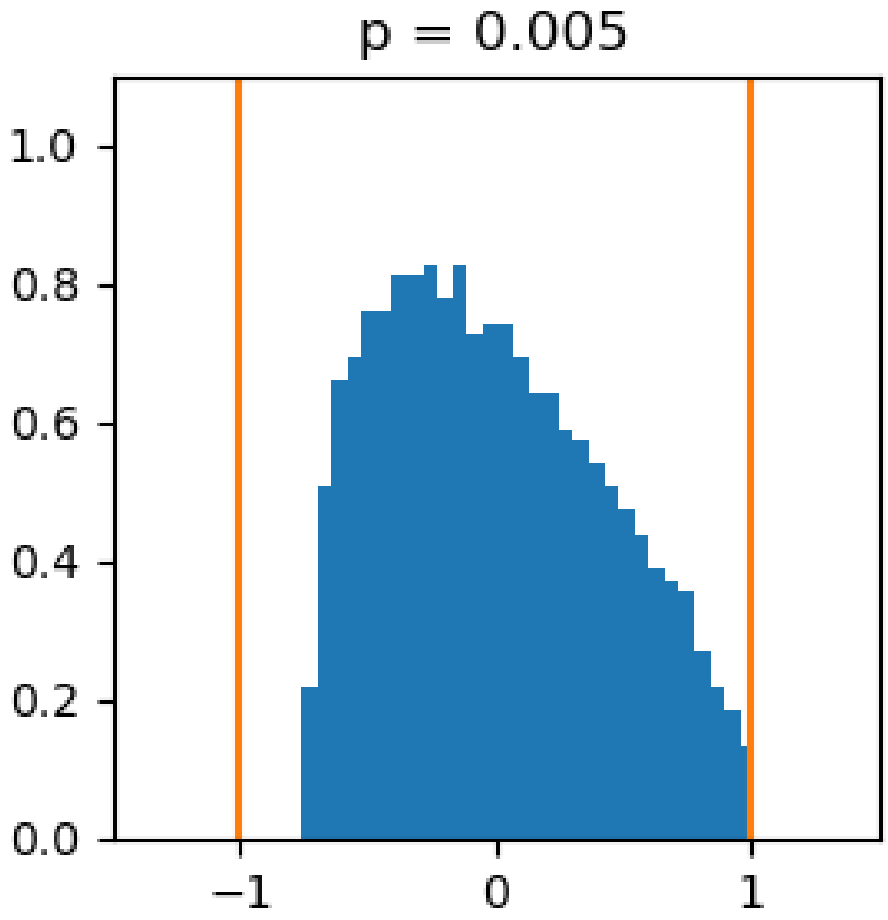




$$\frac{\mathbf{X}\mathbf{X}^{\top}}{\left(2\sqrt{nm}+n+m\right)\sigma^{2}}$$



$$\frac{\mathbf{X}\mathbf{X}^{\top}\odot\mathbf{M}}{\left(2\sqrt{nm}+n\right)\sigma^{2}}$$



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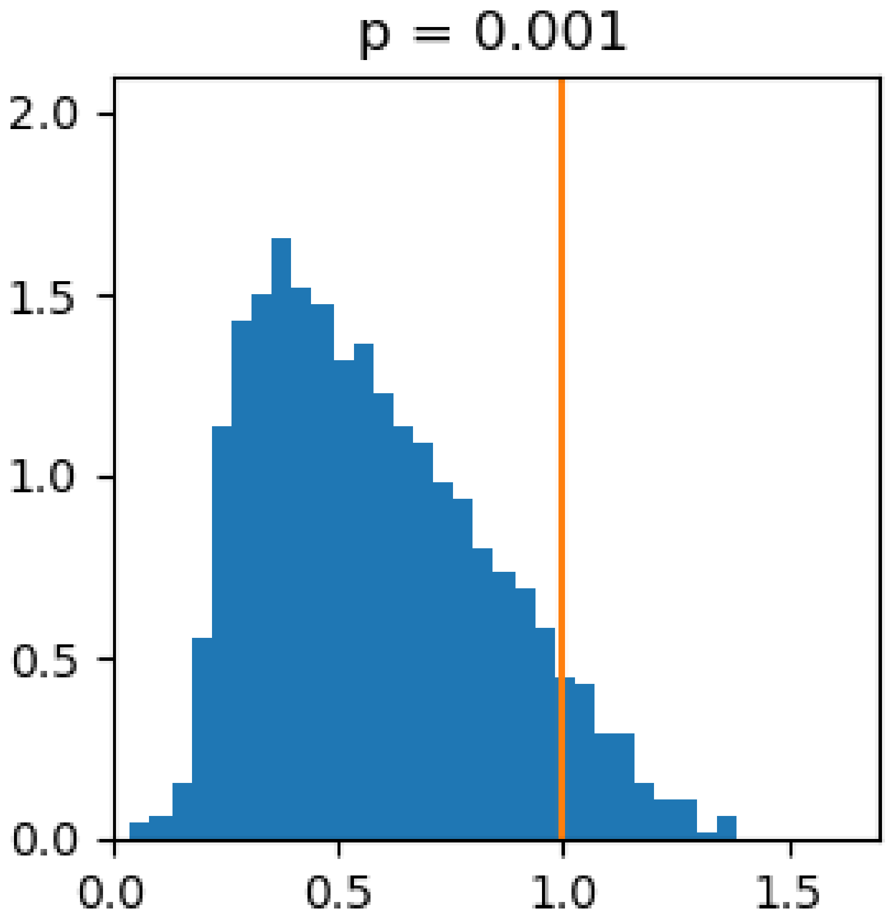

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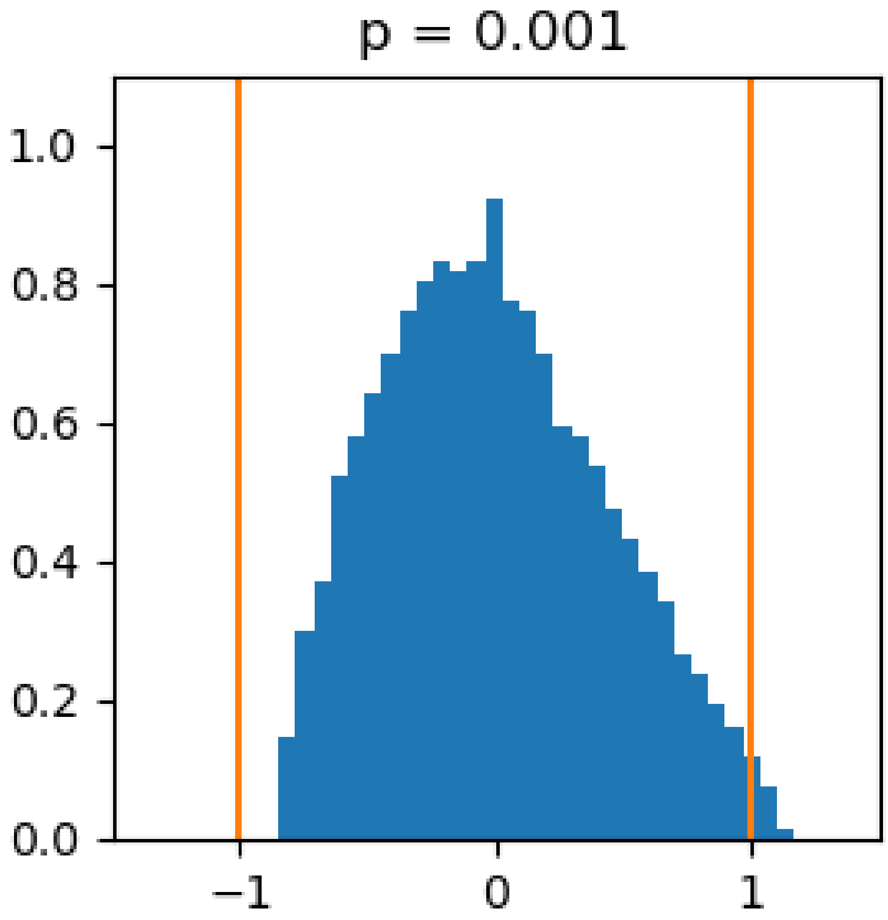

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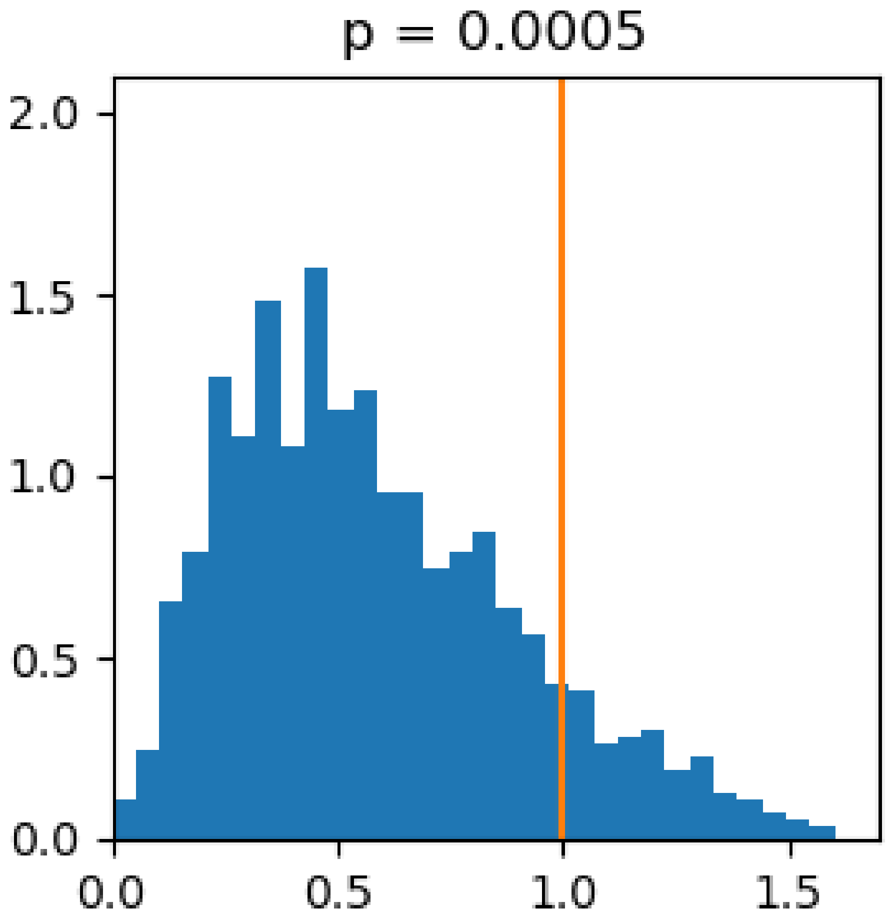

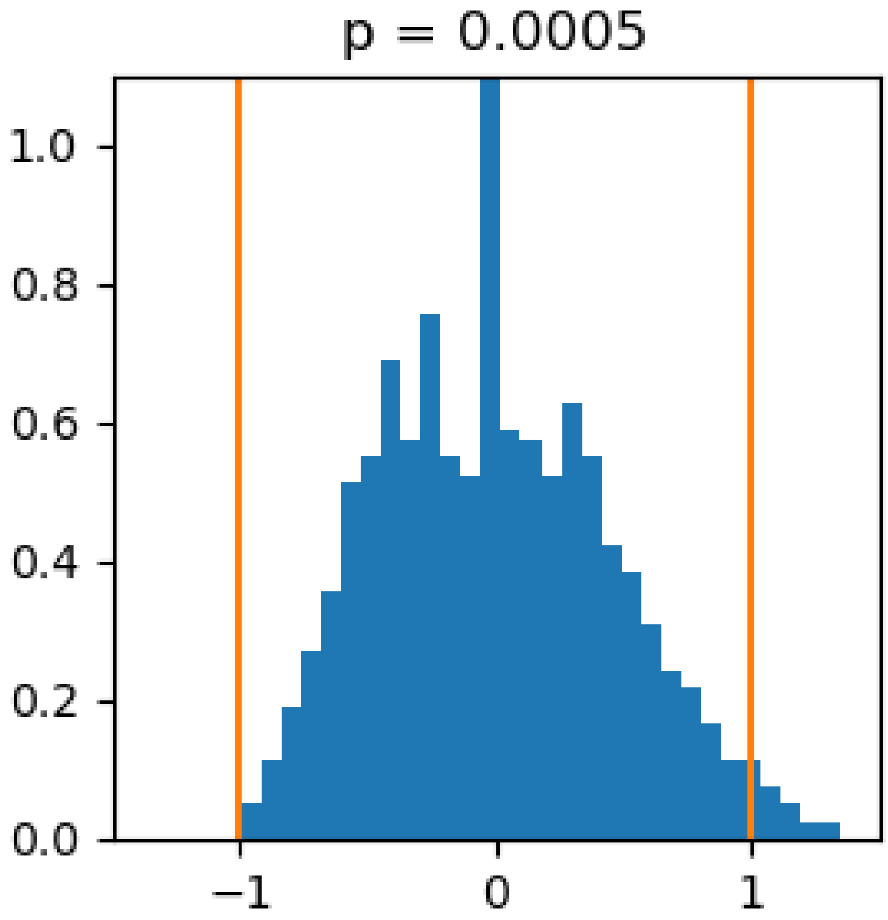

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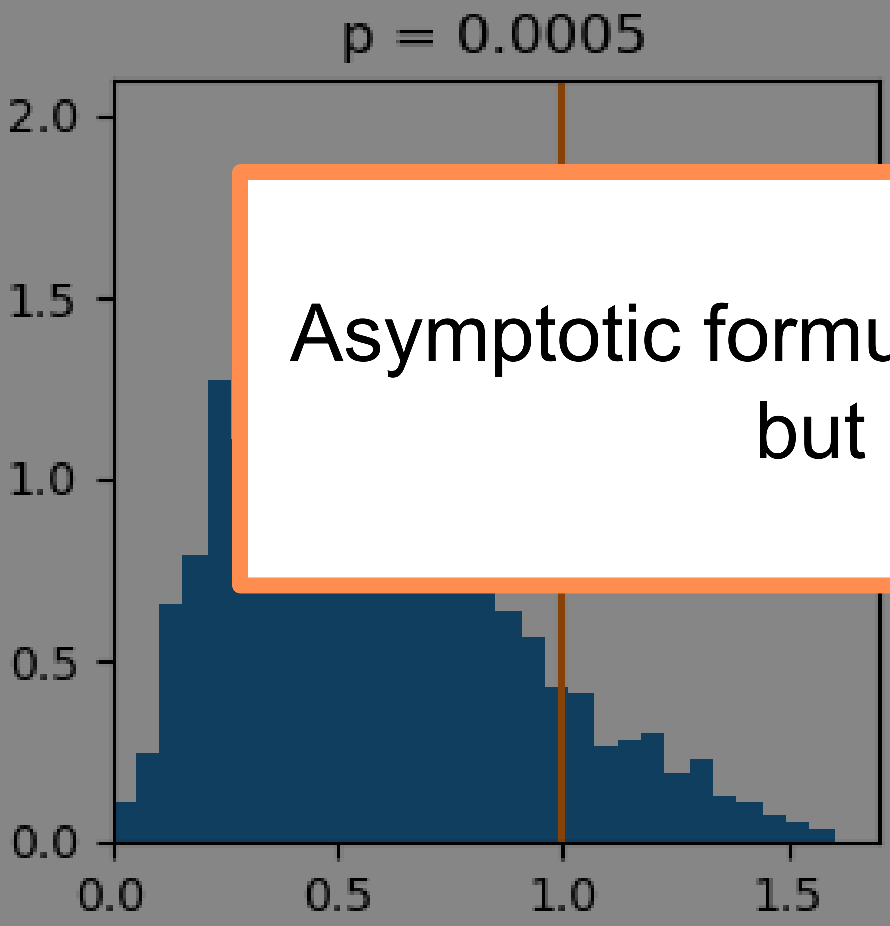

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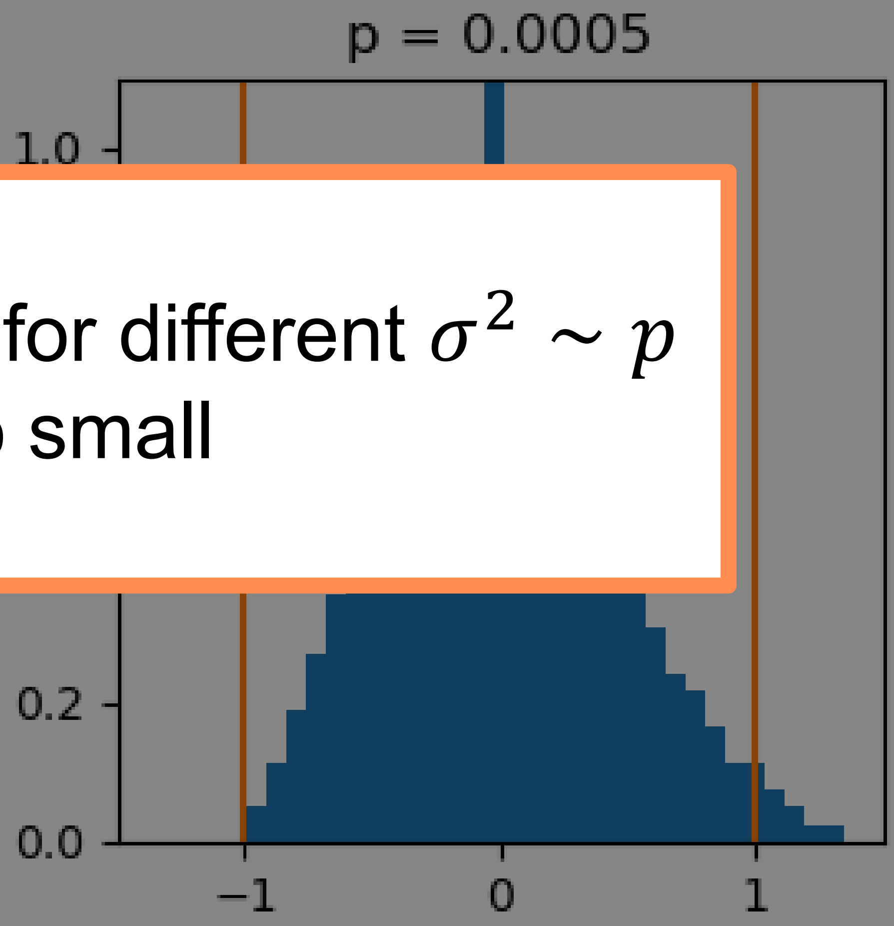

Asymptotic formulas holds for different $\sigma^2 \sim p$
but not for too small

Bound for $||\mathbf{X}\mathbf{X}^\top||$:

Bound for $||\mathbf{X}\mathbf{X}^\top \odot \mathbf{M}||$:

Bound for $||\mathbf{X}\mathbf{X}^\top||$:

$$\left(2\sqrt{nm} + n + m\right)\sigma^2\left(1 + o_{\mathbb{P}}(1)\right)$$

assuming

- $n \asymp m \gg 1$
- $(\log m)/m \ll \sigma^2 \ll m^{-0.8}$

Bound for $||\mathbf{X}\mathbf{X}^\top \odot \mathbf{M}||$:

I. Dumitriu and Y. Zhu. Extreme singular values of inhomogeneous sparse random rectangular matrices. Bernoulli, 30(4), 2024.

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assuming

- $n, m \gg 1$
- $(\log n)/\sqrt{nm} \ll \sigma^2$, or
- $\sqrt{(\log n)/nm} \ll \sigma^2 \ll 1/n$ (?)

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Bound for $||\mathbf{X}\mathbf{X}^\top||$:

$$(2\sqrt{nm} + n + \boldsymbol{m})\sigma^2(1 + o_{\mathbb{P}}(1))$$

Bound for $||\mathbf{X}\mathbf{X}^\top \odot \mathbf{M}||$:

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- Removing the diagonal can make the noise smaller by a factor $O\left(\sqrt{m/n}\right)$
- Significant when $m \gg n$ (wide $\mathbf{X}$), e.g. a flattened d-way $n \times \cdots \times n$-array is $n \times m$ with $m = n^{d-1}$

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For wide $\mathbf{X}$, diagonal and off-diagonal parts of $\mathbf{X}\mathbf{X}^\top$ differ by orders of magnitude

A!

Trace method

E. P. Wigner. Characteristic Vectors of Bordered Matrices with Infinite Dimensions. The Annals of Mathematics, 62(3):548, 1955.

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Symmetric $n \times n$-matrix A and $k \gg \log n$ even:

$$\lambda_{\max}(A)^k \quad \leq \quad \sum_{i=1}^{n} \lambda_i(A)^k \quad \leq \quad n \cdot \lambda_{\max}(A)^k$$

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$$\lambda_{\max}(A)^k \leq \operatorname{tr} A^k \leq n \cdot \lambda_{\max}(A)^k$$

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$$\lambda_{\max}(A)^k \leq \operatorname{tr} A^k \leq n \cdot \lambda_{\max}(A)^k$$

$$\lambda_{\max}(A) \leq \left(\operatorname{tr} A^k\right)^{\frac{1}{k}} \leq e^{\frac{\log n}{k}} \cdot \lambda_{\max}(A)$$

A!

Trace method

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Symmetric $n \times n$-matrix A and $k \gg \log n$ even:

$$\left(\operatorname{tr} A^k\right)^{\frac{1}{k}} \sim \lambda_{\max}(A)$$

Trace method

E. P. Wigner. Characteristic Vectors of Bordered Matrices with Infinite Dimensions. The Annals of Mathematics, 62(3):548, 1955.

Symmetric $n \times n$-matrix A and $k \gg \log n$ even:

A sharp bound for the largest eigenvalue can be found by bounding $\mathbb{E} \operatorname{tr} A^k$ with $k \gg \log n$

A!

Trace method: history (if time)

Bounded k: E. P. Wigner. Characteristic Vectors of Bordered Matrices with Infinite Dimensions. The Annals of Mathematics, 62(3), 1955.

Large k: Z. Füredi, and J. Komlós. The eigenvalues of random symmetric matrices. Combinatorica, 1, 1981.

Large k: Y. G. Sinai, and A. B. Soshnikov. A refinement of Wigner's semicircle law in a neighborhood of the spectrum edge for random symmetric matrices. Functional Analysis and its Applications, 32(2), 1998.

Sparse: A. Khorunzhy. Sparse random matrices: spectral edge and statistics of rooted trees. Advances in Applied Probability, 33(1), 2001.

Sparse: V. H. Vu. Spectral norm of random matrices. Proceedings of the thirty-seventh annual ACM symposium on Theory of computing, 2005.

Trace method: history (if time)

Improvement via Ihara–Bass and nonbacktracking walks: F. Benaych-Georges, C. Bordenave, and A. Knowles. Spectral radii of sparse random matrices. Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, 56(3), 2020.

Improvement via comparison to Gaussian: R. Latała, R. van Handel, and P. Youssef. The dimension-free structure of nonhomogeneous random matrices. Inventiones mathematicae, 214, 2018.

Rectangular matrices via nonbacktracking walks: I. Dumitriu and Y. Zhu. Extreme singular values of inhomogeneous sparse random rectangular matrices. Bernoulli, 30(4), 2024.

Theorem

Let $\mathbf{X} \in [-1, +1]^{n \times m}$ be a random matrix with independent mean-zero entries. Define variance quantities

$$\sigma_{nm}^4 = \frac{1}{nm} \max_i \sum_{i',j} \mathbb{E}X_{ij}^2 \mathbb{E}X_{i'j}^2 ,$$

$$\sigma_m^4 = \frac{1}{m} \max_{i_1,i_2} \sum_j \mathbb{E}X_{i_1 j}^2 \mathbb{E}X_{i_2 j}^2 ,$$

$$\sigma_n^2 = \frac{1}{n} \max_j \sum_i \mathbb{E}X_{ij}^2$$

$$\sigma_{\max}^2 = \max_{i,j} \mathbb{E}X_{ij}^2 .$$

Theorem

Assume $\sqrt{nm\sigma_{nm}^4 + k} \geq n\sigma_n^2 + k$. Then $\mathbb{E}\,\mathrm{tr}\left[(\mathbf{XX}^\top \odot \mathbf{M})^{\mathrm{k}}\right]$ is bounded from above by

$$nke^a\left(\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k} + \frac{8k}{3W\left(\frac{2k}{a}\right)}\right)^{2k},$$

where $a = \frac{3}{e}\left(8e^4k^5\left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2}\right)\right)^{\frac{1}{3}}$.

Proof

Proof

$$\mathbb{E}||\mathbf{X}\mathbf{X}^\top \odot \mathbf{M}||^k \leq \mathbb{E}\,\mathrm{tr}(\mathbf{X}\mathbf{X}^\top \odot \mathbf{M})^k$$

$$= \mathbb{E} \sum_{i_1,\dots,i_k} (\mathbf{X}\mathbf{X}^\top \odot \mathbf{M})_{i_1 i_2} \dots (\mathbf{X}\mathbf{X}^\top \odot \mathbf{M})_{i_{k-1} i_k} (\mathbf{X}\mathbf{X}^\top \odot \mathbf{M})_{i_k i_1}$$

$$= \mathbb{E} \sum_{\substack{i_1 \neq \dots \neq i_k, \\ j_1,\dots,j_k}} X_{i_1 j_1} X^\top_{j_1 i_2} X_{i_2 j_2} X^\top_{j_2 i_3} \dots X_{i_{k-1} j_{k-1}} X^\top_{j_{k-1} i_k} X_{i_k j_k} X^\top_{j_k i_1}$$

$$= \mathbb{E} \sum_{\substack{i_1 \neq \dots \neq i_k, \\ j_1,\dots,j_k}} \prod_e X_e^{\#\{\text{walk } i_1 j_1 i_2 j_2 \dots i_{k-1} j_{k-1} i_k j_k i_1 \text{ traverses through } e\}}$$

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$$\mathbb{E}||\mathbf{X}\mathbf{X}^\top \odot \mathbf{M}||^k$$
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The walk $i_1 j_1 \ldots i_k j_k i_1$ is a graph homomorphism $f: C \to G$, where

- C is the cycle graph with vertex set $[2k]$ and the rule is defined by $f(1) \ldots f(2K) = i_1 j_1 \ldots i_k j_k$;
- f is surjective, i.e, $V(G) = \{i_1, j_1, \ldots, i_k, j_k\} \subset [n] \sqcup [m]$
- G is coinduced, i.e., $E(G) = \left\{\{f(1), f(2)\}, \ldots, \{f(2k), f(1)\}\right\}$

Proof

$$\mathbb{E}||\mathbf{X}\mathbf{X}^\top \odot \mathbf{M}||^k \leq \mathbb{E} \sum_{f:C\to G} \prod_{e\in E(G)} X_e^{\#\{i\,:\,\{f(i),f(i+1)\}=e\}},$$

where we sum over those f for which

- $f(i) \in [n]$ for odd i, $f(j) \in [m]$ for even j.
- $f(j-1) \neq f(j+1)$ for all even j (nonbacktracking at m-nodes)
- G is coinduced by f and f is surjective

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Proof

Divide the complex task $\mathbb{E}\sum_f \prod_{e\in E(G)} X_e^2$ into parts:

Task 0: Every f admits a decomposition $f_3 \circ f_2 \circ f_1$

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Task 2: How many f_2?

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Task 1: How many f_1?

Task 2: How many f_2?

Task 3: How to bound $\mathbb{E}\sum_{f_3} \prod_{e\in E(G)} X_e^2$?

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Task 0: Every f admits a decomposition $f_3 \circ f_2 \circ f_1$

Task 1: How many f_1?

Task 2: How many f_2?

Task 3: How to bound $\mathbb{E}\sum_{f_3} \prod_{e\in E(G)} X_e^2$?

Task 4: Collect everything together

Proof

Divide the complex task $\mathbb{E}\sum_f \prod_{e\in E(G)} X_e^2$ into parts:

Task 0: Every f admits a decomposition $f_3 \circ f_2 \circ f_1$ — **Category theory**

Task 1: How many f_1? — **Combinatorics**

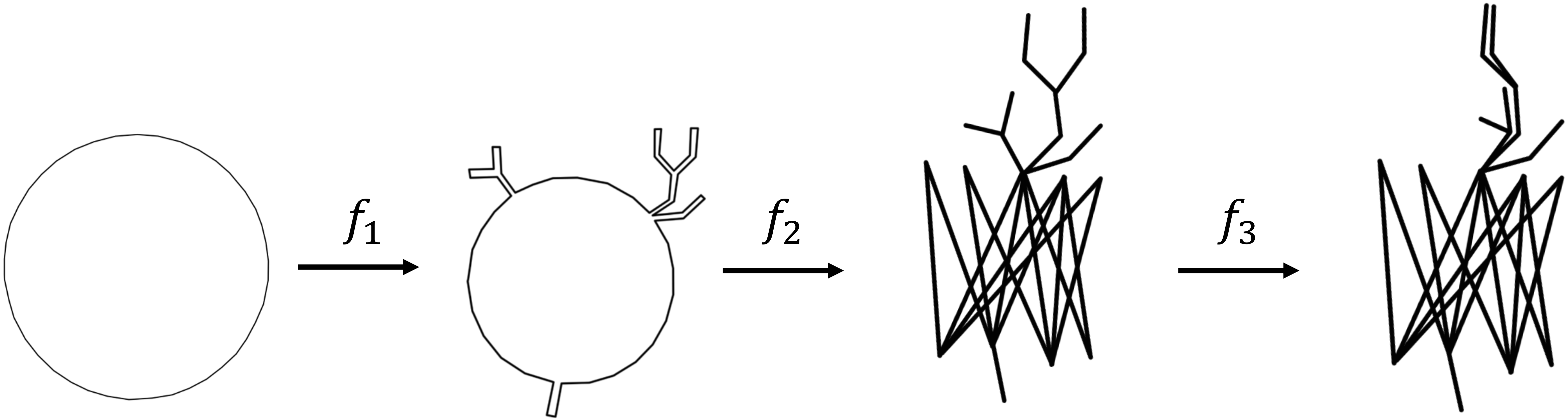
Task 2: How many f_2? — **Combinatorics**

Task 3: How to bound $\mathbb{E}\sum_{f_3} \prod_{e\in E(G)} X_e^2$? — **Hölder's inequalities**

Task 4: Collect everything together — **Calculations**

A!

Proof: Task 0



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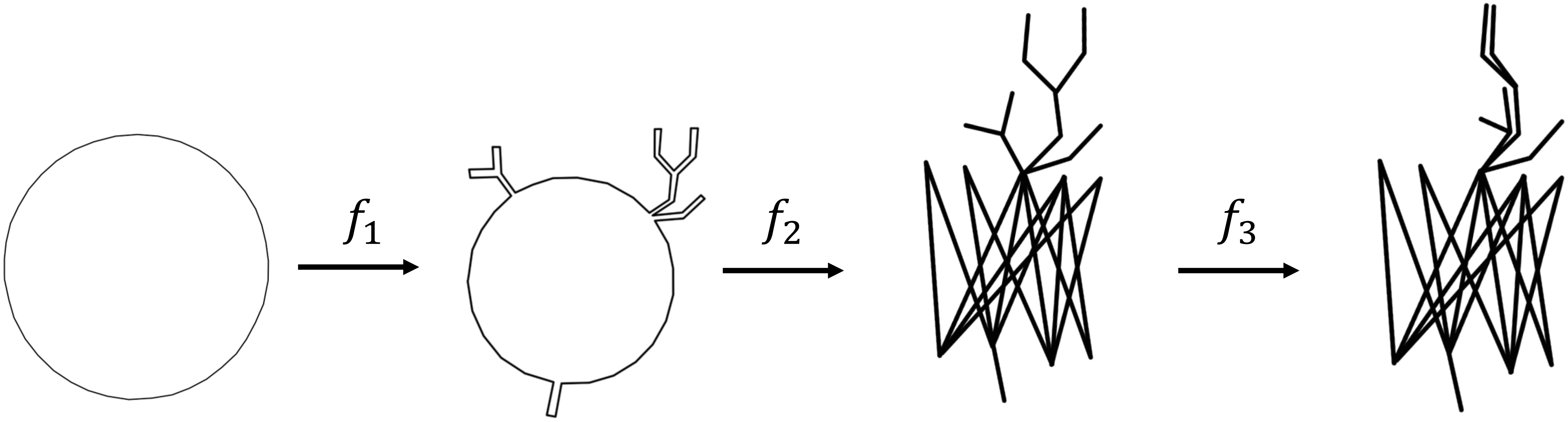
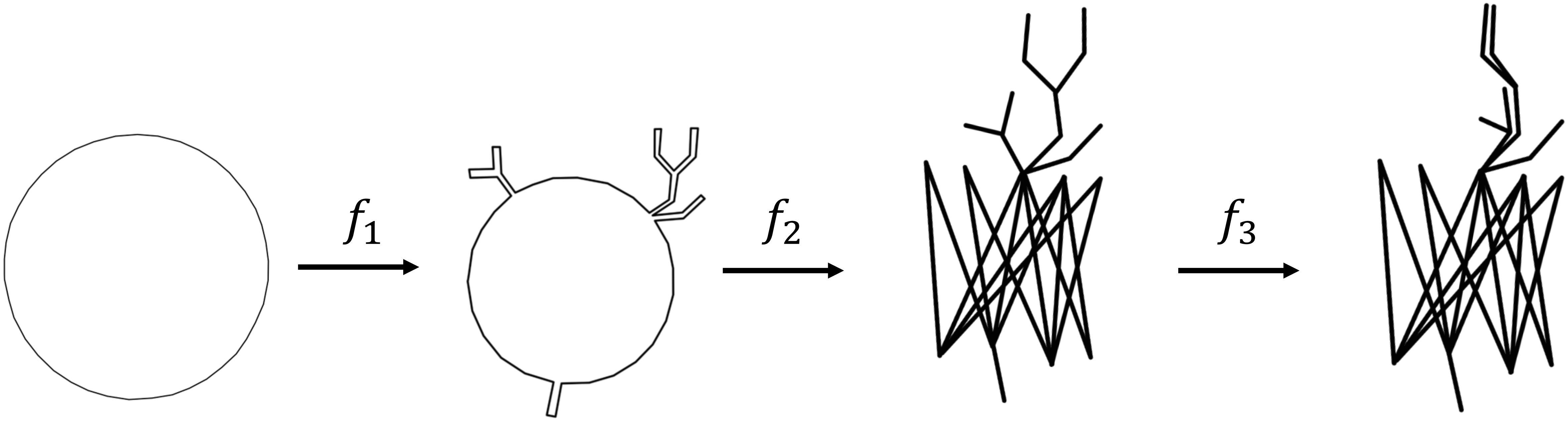


Fold trees only, edges on trees traversed twice

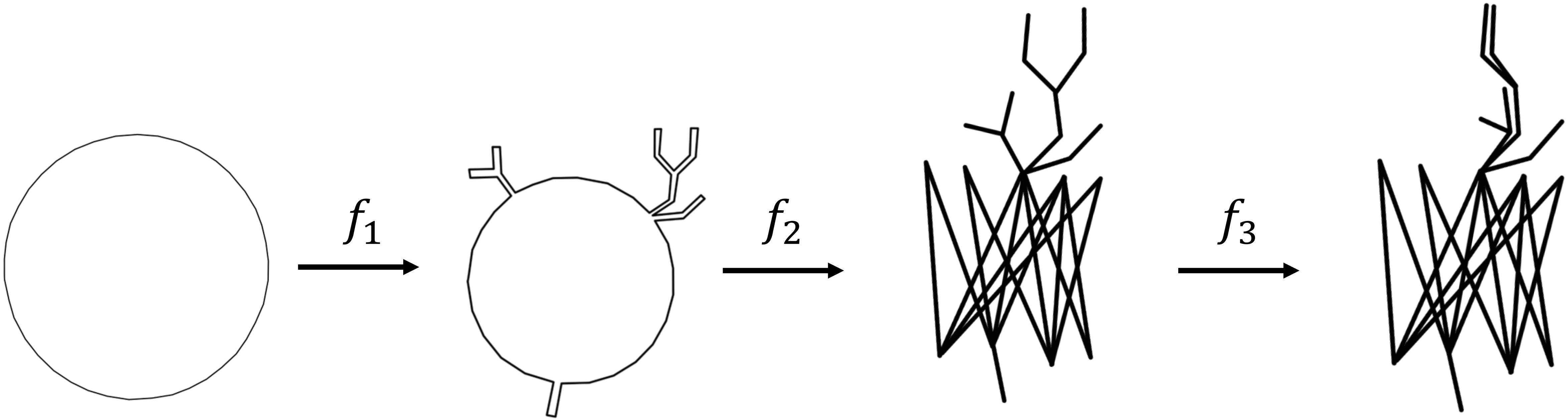
Proof: Task 0



Fold trees only, edges on trees traversed twice

Fold the cycle into a 2-core (no 1-degree nodes) only, edges covered at least twice

Proof: Task 0



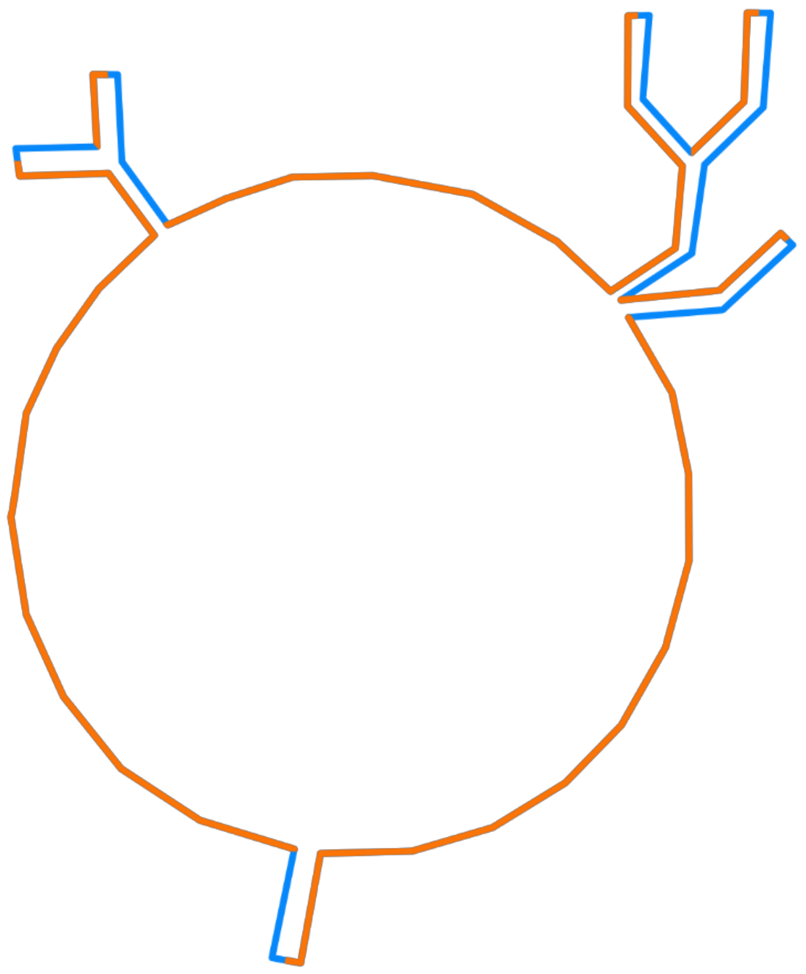
Fold trees only, edges on trees traversed twice

Fold the cycle into a 2-core (no 1-degree nodes) only, edges covered at least twice

Fold trees such that the distance to the core is preserved

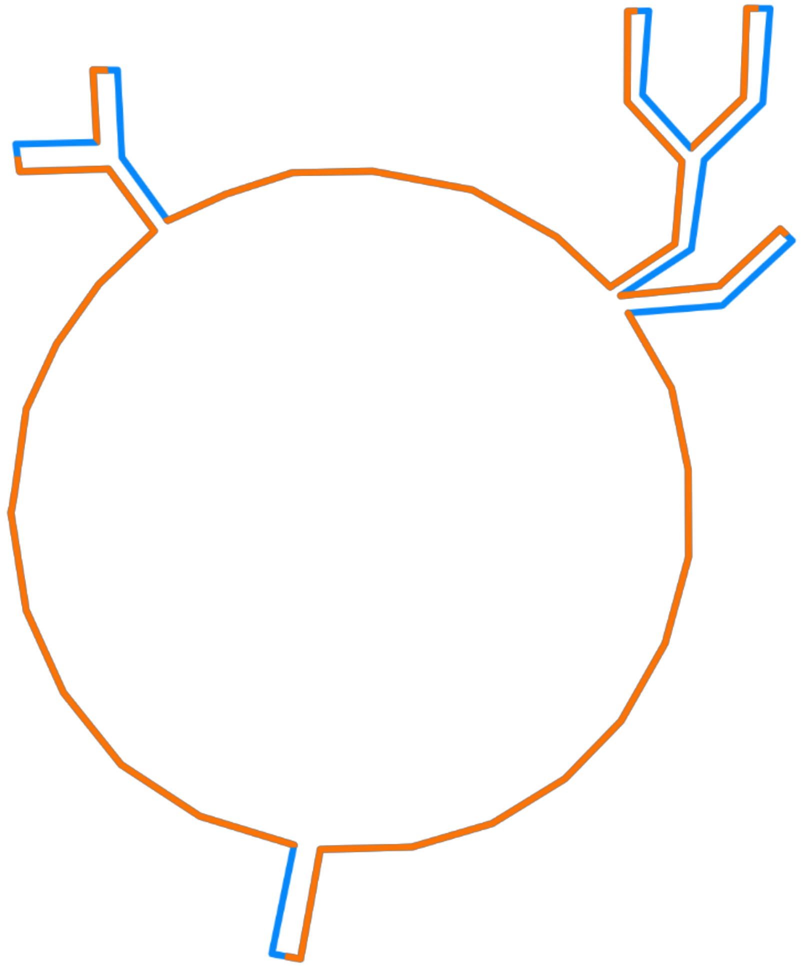
Proof: Task 1

- Mark "–" if traversed for the second time ⟺ step towards the cycle;
- Mark "+" if traversed for the first time ⟺ step away from the cycle



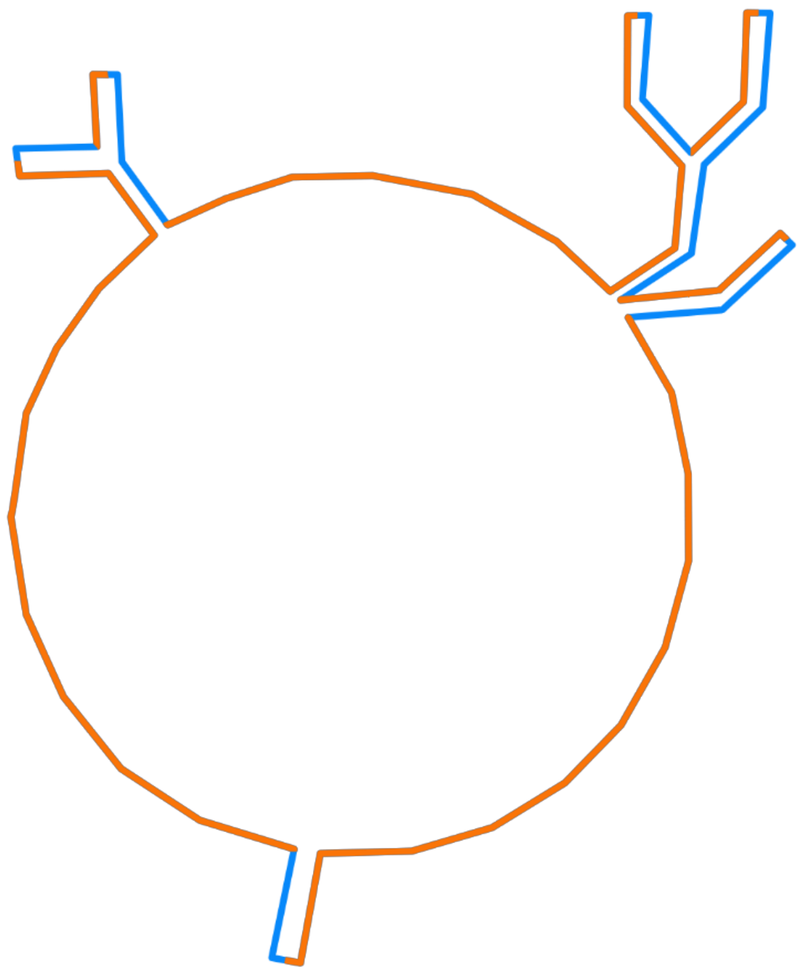
Proof: Task 1

- How many f_1? $\leq 2^{2k}$



Proof: Task 1

- How many f_1? $\leq 2^{2k}$
- Improvement $\leq 3^k$: two consecutive steps (n-m-n) can be $+\,+$, $-\,-$ or $-\,+$ but not $+\,-$ because the walk is nonbacktracking at m-nodes



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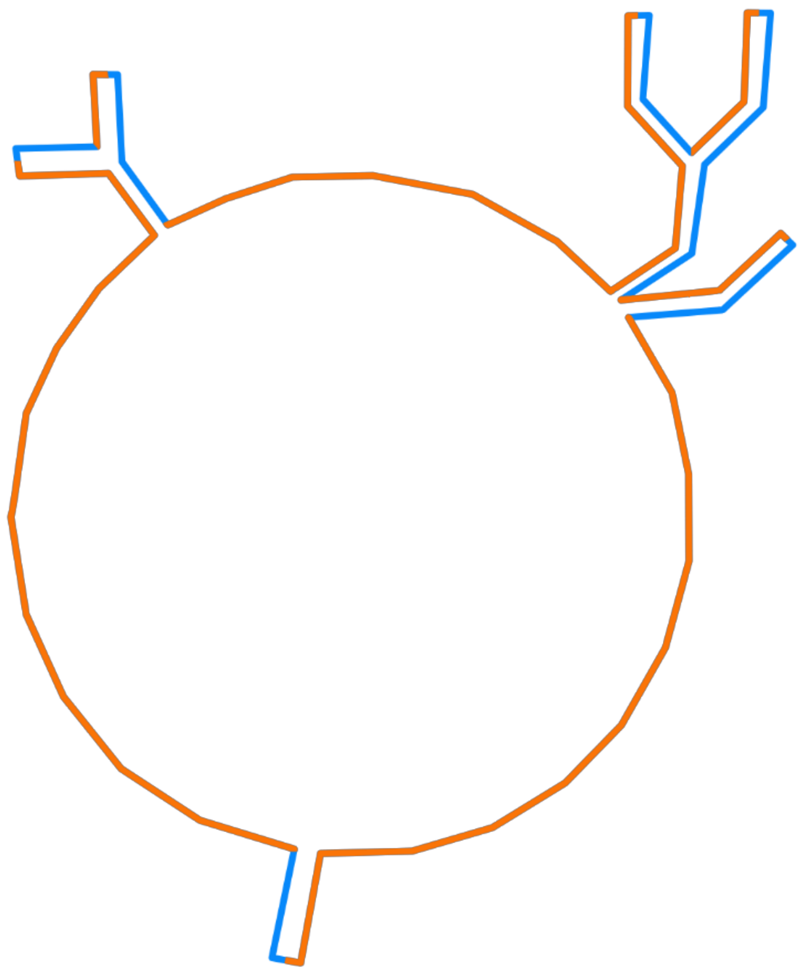
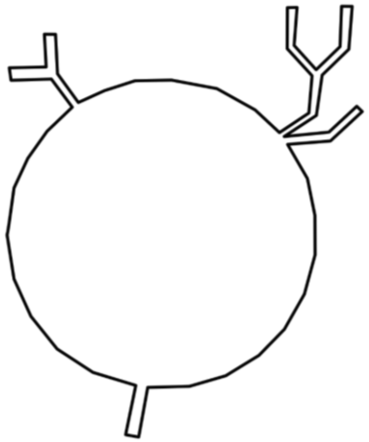
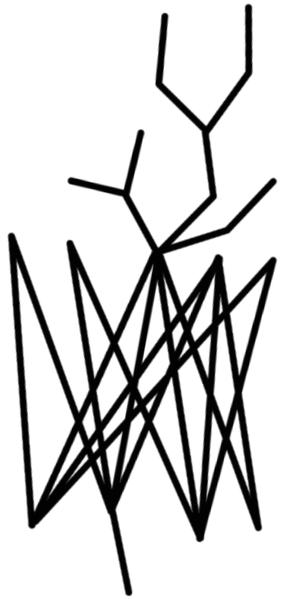
- Improvement $\leq \binom{k}{\#\{-+\}} 2^{k-\#\{-+\}}$: track how many times $-\,+$ is chosen ⇔ how many n-nodes are there more than m-nodes



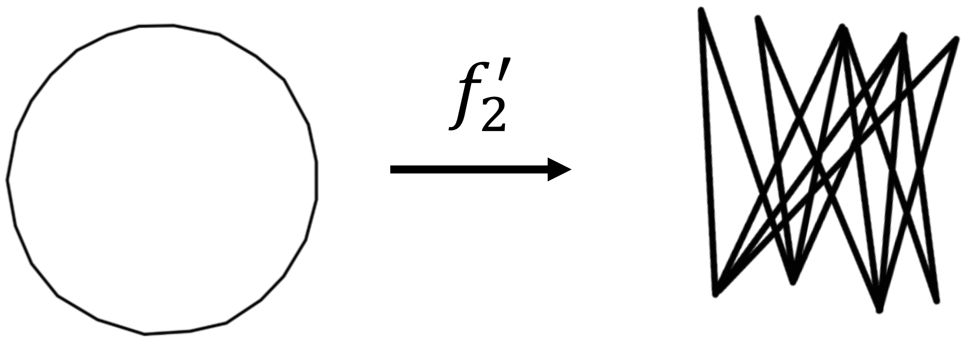
Proof: Task 2



f_2



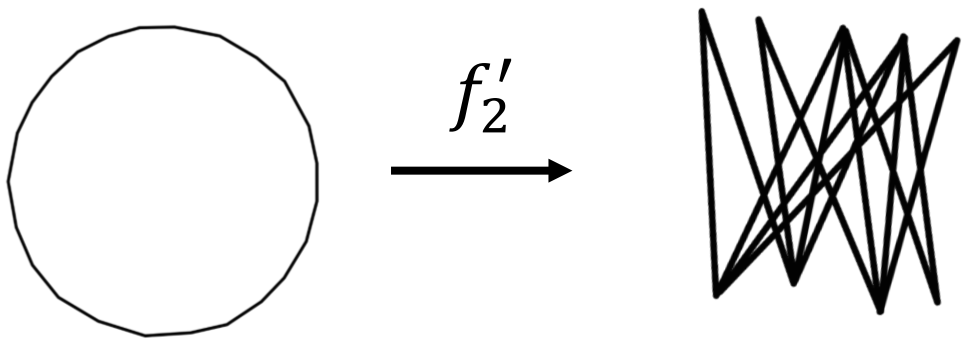
Proof: Task 2



Proof: Task 2

How many $f_2': C_{2k'} \to G$ such that

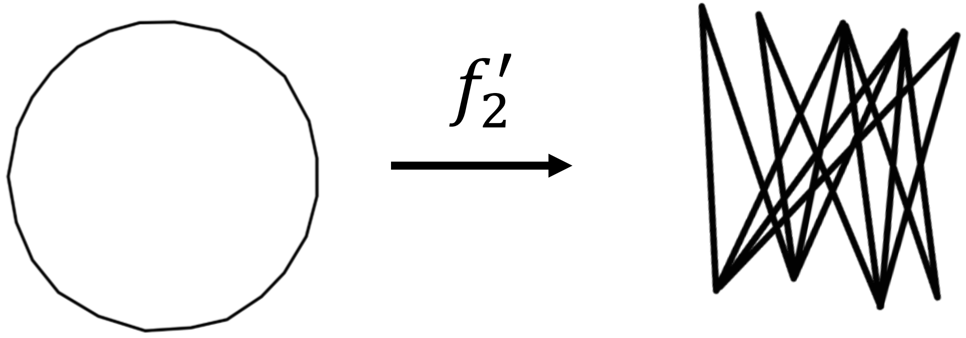
- f_2' surjective and G is coinduced (w.l.g. G is a quotient graph and f_2 is the corresponding natural projection),
- every edge $e \in E(G)$ is covered at least twice,
- Edge count $|E(G)| = l$ and genus $|E(G)| - |V(G)| + 1 = g$ are fixed



Proof: Task 2

Idea: Choose one step $f_2(i)f_2(i+1) \Leftrightarrow$ one node $f_2(i+1)$ at a time.

- Type (i) step = the edge is traversed for the first time
- Type (ii) step = the edge is traversed neither for the first nor the last time
- Type (iii) step = the edge is traversed for the last time



	Type (i)			Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$			$\Delta g = -1$	$\Delta g = 0$
		$\deg \leq 1$ prior to the step	$\deg \geq 2$ prior to the step			
How many?	g	$l - g$		$2k' - 2l$	g	$l - g$
Combinatorial cost	k'	1		Max deg	Max deg	1
Choose types	$\binom{2k'}{g}$		$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

	Type (i)			Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$			$\Delta g = -1$	$\Delta g = 0$
		$\deg \leq 1$ prior to the step	$\deg \geq 2$ prior to the step			
How many?	g	$l - g$		$2k' - 2l$	g	$l - g$
Combinatorial cost	k'				Max deg	1
Choose types	$\binom{2k'}{g}$				$\binom{2k'}{g}$	

Genus increases g times

	Type (i)			Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$			$\Delta g = -1$	$\Delta g = 0$
		$\deg \leq 1$ prior to the step	$\deg \geq 2$ prior to the step			
How many?	g	$l - g$		$2k' - 2l$	g	$l - g$
Combinatorial cost	k'				Max deg	1
Choose types	$\binom{2k'}{g}$				$\binom{2k'}{g}$	

Every edge is traversed once for the first/last time

	Type (i)			Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$			$\Delta g = -1$	$\Delta g = 0$
		$\deg \leq 1$ prior to the step	$\deg \geq 2$ prior to the step			
How many?	g	$l - g$		$2k' - 2l$	g	$l - g$
Combinatorial cost	k'				Max deg	1
Choose types	$\binom{2k'}{g}$				$\binom{2k'}{g}$	

$2k$ steps in total

	Type (i)		Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$		$\Delta g = -1$	$\Delta g = 0$
How many?	g	$l - g$	$2k' - 2l$	g	$l - g$
Combinatorial cost	k'	1	Max deg	Max deg	1
Choose types	$\binom{2k'}{g}$	$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

Number of nodes $=$

$$l + 1 - g \leq l \leq k$$

	Type (i)		Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$		$\Delta g = -1$	$\Delta g = 0$
How many?	g	$l - g$	$2k' - 2l$	g	$l - g$
Combinatorial cost	k'	1	Max deg	Max deg	1
Choose types	$\binom{2k'}{g}$	$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

New edge and new node

	Type (i)		Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$		$\Delta g = -1$	$\Delta g = 0$
How many?	g	$l - g$	$2k' - 2l$	g	$l - g$
Combinatorial cost	k'	1	Max deg	Max deg	1
Choose types	$\binom{2k'}{g}$	$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

Choose neighbor node

	Type (i)		Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$		$\Delta g = -1$	$\Delta g = 0$
How many?	g	$l - g$	$2k' - 2l$	g	$l - g$
Combinatorial cost	k'	1	$2g$	$2g$	1
Choose types	$\binom{2k'}{g}$	$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

Fact: $\deg \leq 2g$

$\Delta g = 0$: In the final graph but without traversed type (iii) steps/edges

The step deletes both the source node and the edge. Since the source node is no longer visited, its degree must be 1 prior to the step

	Type (i)			Type (ii)	Type (iii)	
					$\Delta g = -1$	$\Delta g = 0$
H					g	$l - g$
Co					$2g$	1
Choose types	$\binom{2k'}{g}$		$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

	Type (i)			Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$			$\Delta g = -1$	$\Delta g = 0$
		$\mathbf{deg} \leq 1$ prior to the step				
How many?	g	$l - g$		$2k' - 2l$	g	$l - g$
Combinatorial cost	k'	1		$2g$	$2g$	1
Choose types	$\binom{2k'}{g}$		$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

Evident

	Type (i)		Type (ii)	Type (iii)	
		$\Delta g = 0$		$\Delta g = -1$	$\Delta g = 0$
				g	$l - g$
cost	k	1	$2g$	$2g$	1
Choose types	$\binom{2k'}{g}$	$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

$$\sum_{x \in V} (\deg x - 2) = 2|E| - 2|V| \leq 2g,$$

$$\sum_{s \leq 2g} \binom{2k}{2g} \leq \left(\frac{2ek}{2g}\right)^{2g} = \left(\frac{ek}{g}\right)^{2g}$$

	Type (i)			Type (ii)	Type (iii)	
	$\Delta g = +1$	$\Delta g = 0$				$= 0$
		$\deg \leq 1$ prior to the step	$\deg \geq 2$ prior to the step			
How many?	g	$l - g$		$2k' - 2l$	g	$l - g$
Combinatorial cost	k'	1		$2g$	$2g$	1
Choose types	$\binom{2k'}{g}$		$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

If the degree is ≥ 2, it cannot be type (i).

	Type (i)			Type (ii)	Type (iii)	
						$\Delta g = 0$
						$l - g$
cost	k	1		$2g$	$2g$	1
Choose types	$\binom{2k'}{g}$		$\left(\frac{ek'}{g}\right)^{2g}$	$\binom{2k'}{2l}$	$\binom{2k'}{g}$	

If the degree is one and the type is (iii), then a degree-one node is created. Since $\Delta g = 0$, that node is not visited later and hence the degree-one node remains in the final graph (CONTRADICTION)

<table>
<tr><th rowspan="3"></th><th colspan="3">Type (i)</th><th rowspan="3">Type (ii)</th><th colspan="2">Type (iii)</th></tr>
<tr><th rowspan="2">$\Delta g = +1$</th><th colspan="2">$\Delta g = 0$</th><th rowspan="2">$\Delta g = -1$</th><th rowspan="2">$\Delta g = 0$</th></tr>
<tr><th>$\deg \leq 1$ prior to the step</th><th>$\deg \geq 2$ prior to the step</th></tr>
<tr><td>How many?</td><td>g</td><td colspan="2">$l - g$</td><td>$2k' - 2l$</td><td>g</td><td>$l - g$</td></tr>
<tr><td>Combinatorial cost</td><td>k'</td><td colspan="2">1</td><td>$2g$</td><td>$2g$</td><td>1</td></tr>
<tr><td>Choose types</td><td>$\binom{2k'}{g}$</td><td></td><td>$\left(\frac{ek'}{g}\right)^{2g}$</td><td>$\binom{2k'}{2l}$</td><td>$\binom{2k'}{g}$</td><td></td></tr>
</table>

	Type (i)		Type (ii)	Type (iii)	
How m…					
Combin… co…					
Choose types	$\binom{2k}{g}$	$\left(\frac{ek}{g}\right)$	$\binom{2k}{2l}$	$\binom{2k}{g}$	

Task 2: How many f_2?

$$\leq \binom{2k'}{2l}\binom{2k'}{g}^{2}\left(\frac{ek'}{g}\right)^{2g} k'^{g}(2g)^{2k'-2l}(2g)^{g}$$

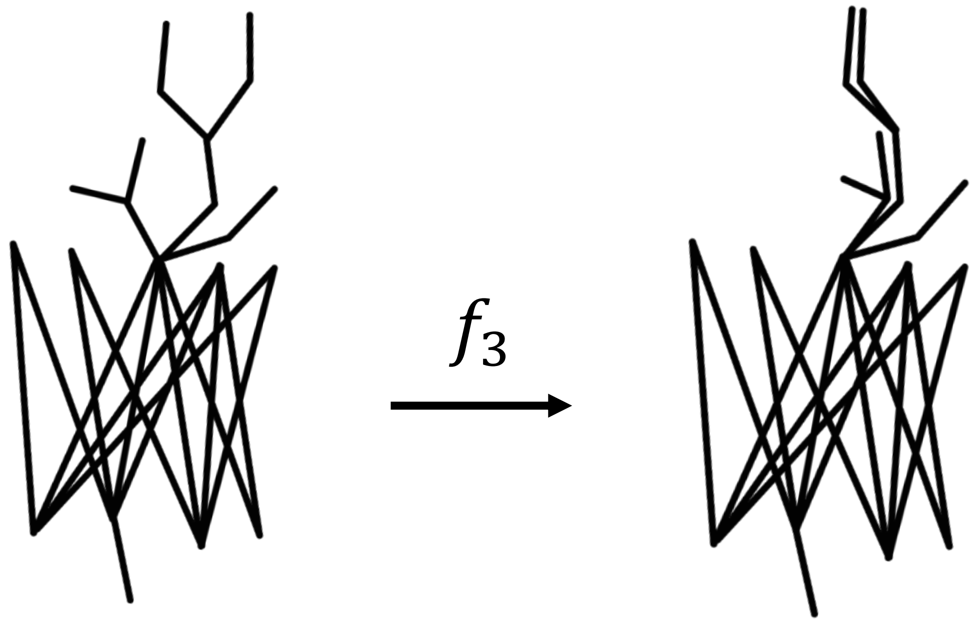
$$\leq \binom{2k'}{2l}\left(\frac{8e^{4}k^{5}}{g^{3}}\right)^{g}(2g)^{2k'-2l}$$

	Type (i)			Type (ii)	Type (iii)	
How m...						g
Combin... co...						
Choose types	$\binom{\cdot}{g}$		$\binom{\cdot}{g}$	$\binom{\cdot}{2l}$	$\binom{\cdot}{g}$	

Task 2: How many f_2?

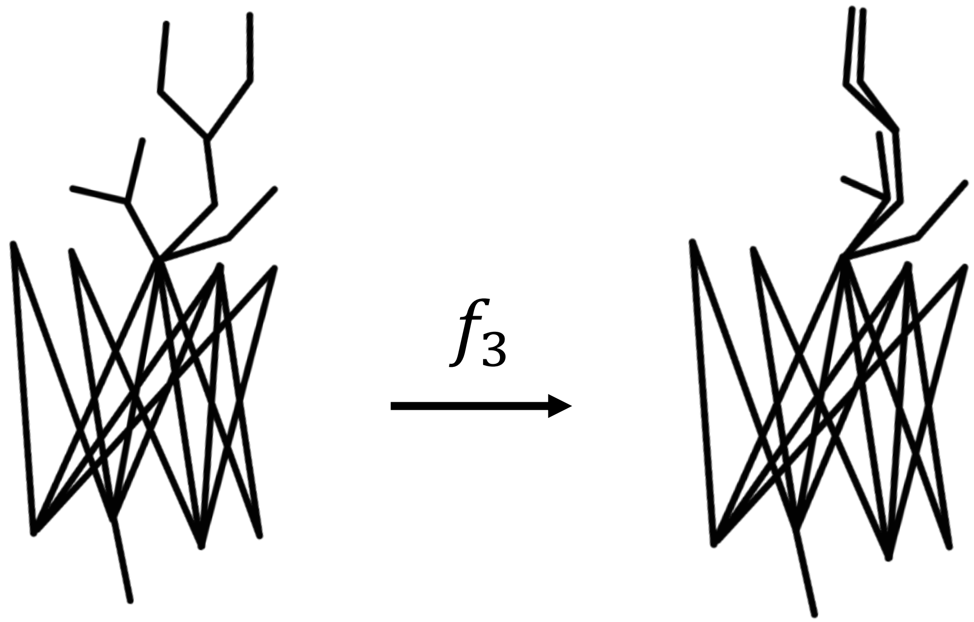
Bound is larger for larger genus

Proof: Task 3



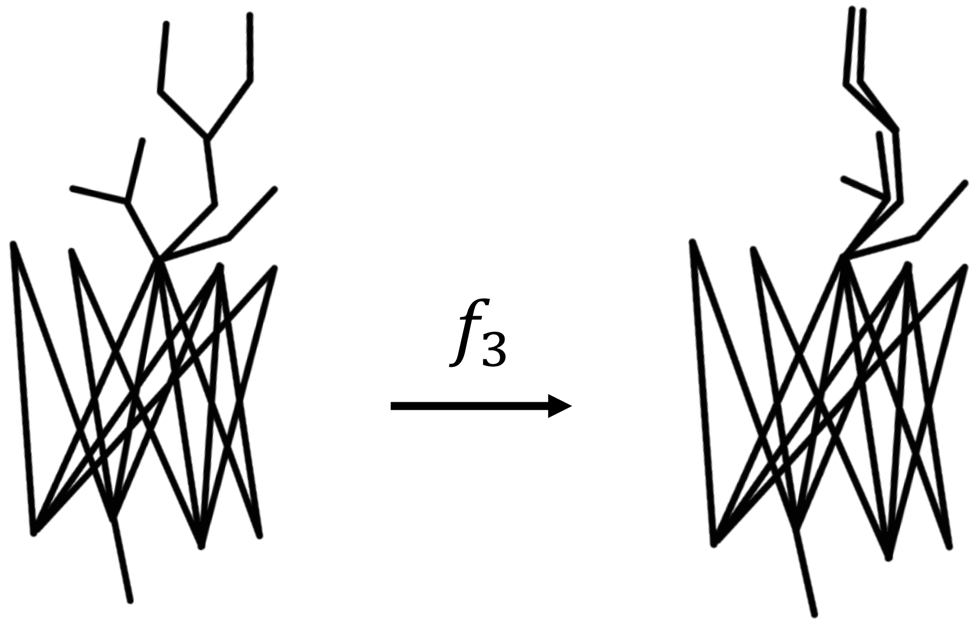
Proof: Task 3

- Fix the source graph of $f_3 \colon H \to G$
- 2-core of H (= 2-core of G) has genus g and l edges



Proof: Task 3

- If there exists an m-node $j \in V(H)$ with exactly two n-node neighbors i_1, i_2 that lie on a cycle, sum over all possible $f_3(j) \in [m]$ and take out the common factor $\sum_{f_3(j)} \mathbb{E}X^2_{f_3(i_1)f_3(j)} \mathbb{E}X^2_{f_3(i_2)f_3(j)} \leq m\sigma_m^4$
- If there exists an n-m-edge $\{i, j\} \in E(H)$ lying on a cycle and i has degree at least 3, take out the common factor $\mathbb{E}X^2_{f_3(i)f_3(j)} \leq \sigma^2_{\max}$



Proof: Task 3

After removing g_1 n-m-n bridges and $g_2 = g - g_1$ edges this way, we are left with a tree $T \subset H$ leaving us the sum

$$(m\sigma_m^4)^{g_1}\sigma_{\max}^{2g_2} \cdot \mathbb{E} \sum_{f_3|_T} \prod_{e \in E(T)} X_{f_3[e]}^2$$

Moreover, no leaf is an m-node in T!

Proof: Task 3

Lemma. For $X_1, \ldots, X_n \geq 0$ and $p_i \geq q_i$,

$$\prod_i \frac{\mathbb{E}X_i^{p_i}}{\mathbb{E}X_i^{q_i}} \leq \max_i \frac{\mathbb{E}X_i^{p_1+\cdots+p_n}}{\mathbb{E}X_i^{q_1+\cdots+q_n}}$$

Proof: Task 3

Lemma.

$$\mathbb{E}\sum_{f_3|_T}\prod_{e\in E(T)} X_{f_3[e]}^2$$

$$\leq n\left(\max_i \mathbb{E}\left(\sum_{i',j:i'\neq i} X_{ij}^2 X_{i'j}^2\right)^{|V_m(T)|}\right)\left(\max_j \frac{\mathbb{E}\left(\sum_i X_{ij}^2\right)^{|V_n(T)|-1}}{\mathbb{E}\left(\sum_i X_{ij}^2\right)^{|V_m(T)|}}\right)$$

$$\leq n(nm\sigma_{nm}^4 + k)^{|V_m(T)|}\left(n\sigma_n^2 + k\right)^{|V_n(T)|-1-|V_m(T)|}$$

$$= n(nm\sigma_{nm}^4 + k)^{|V_m(T)|}\left(n\sigma_n^2 + k\right)^{|V(T)|-1-2|V_m(T)|}$$

Proof: Task 3

$$(m\sigma_m^4)^{g_1}\sigma_{\max}^{2g_2} \cdot n(nm\sigma_{nm}^4 + k)^{|V_m(T)|}\left(n\sigma_n^2 + k\right)^{|V(T)|-1-2|V_m(T)|}$$

Proof: Task 3

$$\leq (m\sigma_m^4)^{g_1}\sigma_{\max}^{2g_2} \cdot n\left(\frac{\sqrt{nm\sigma_{nm}^4 + k}}{n\sigma_n^2 + k}\right)^{2|V_m(T)|}\left(n\sigma_n^2 + k\right)^{|V(T)|-1}$$

Proof: Task 3

$$\leq (m\sigma_m^4)^{g_1}\sigma_{\max}^{2g_2} \cdot n\left(\frac{\sqrt{nm\sigma_{nm}^4 + k}}{n\sigma_n^2 + k}\right)^{2|V_m(T)|} \left(n\sigma_n^2 + k\right)^{|V(T)|-1}$$

Since $|V_n(T)| - |V_m(T)| \geq \#\{-+\} + 1$, the number of m-nodes has an upper bound $|V_m(T)| \leq \frac{|V(T)|-\#\{-+\}-1}{2}$. Here

$$\begin{aligned}|V(T)| &= |V(H)| - g_1 \\ &= |E(H)| - g - g_1 + 1 \\ &= k - k' + l - 2g_1 - g_2 + 1.\end{aligned}$$

Proof: Task 3

$$= (m\sigma_m^4)^{g_1}\sigma_{\max}^{2g_2} \cdot n\left(\frac{\sqrt{nm\sigma_{nm}^4 + k}}{n\sigma_n^2 + k}\right)^{2|V_m(T)|} \left(n\sigma_n^2 + k\right)^{k-k'+l-2g_1-g_2}$$

Since $|V_n(T)| - |V_m(T)| \geq \#\{- +\} + 1$, the number of m-nodes has an upper bound $|V_m(T)| \leq \frac{|V(T)|-\#\{-+\}-1}{2}$. Here

$$\begin{aligned} |V(T)| &= |V(H)| - g_1 \\ &= |E(H)| - g - g_1 + 1 \\ &= k - k' + l - 2g_1 - g_2 + 1. \end{aligned}$$

Proof: Task 3

$$\leq (m\sigma_m^4)^{g_1}\sigma_{\max}^{2g_2} \cdot n\left(\frac{\sqrt{nm\sigma_{nm}^4 + k}}{n\sigma_n^2 + k}\right)^{k-k'+l-2g_1-g_2-\#\{-+\}}$$

$$\times \left(n\sigma_n^2 + k\right)^{k-k'+l-2g_1-g_2}$$

Proof: Task 3

$$\leq (m\sigma_m^4)^{g_1}\sigma_{\max}^{2g_2} \cdot n\sqrt{nm\sigma_{nm}^4 + k}^{\,k-k'+l-2g_1-g_2-\#\{-+\}}$$

$$\times \left(n\sigma_n^2 + k\right)^{\#\{-+\}}$$

Proof: Task 3

$$\leq \left(\frac{m\sigma_m^4}{nm\sigma_{nm}^4 + k}\right)^{g_1} \sigma_{\max}^{2g_2} \cdot n\sqrt{nm\sigma_{nm}^4 + k}^{\,k-k'+l-g_2-\#\{-+\}}$$

$$\times \left(n\sigma_n^2 + k\right)^{\#\{-+\}}$$

Proof: Task 3

$$\leq \left(\frac{\sigma_m^4}{n\sigma_{nm}^4}\right)^{g_1} \sigma_{\max}^{2g_2} \cdot n\sqrt{nm\sigma_{nm}^4 + k}^{\,k-k'+l-g_2-\#\{-+\}}$$

$$\times \left(n\sigma_n^2 + k\right)^{\#\{-+\}}$$

A!

Proof: Task 3

$$\leq \left(\frac{\sigma_m^4}{n\sigma_{nm}^4}\right)^{g_1} \left(\frac{\sigma_{\max}^2}{\sqrt{nm\sigma_{nm}^4 + k}}\right)^{g_2} \cdot n\sqrt{nm\sigma_{nm}^4 + k}^{\,k-k'+l-\#\{-+\}}$$

$$\times \left(n\sigma_n^2 + k\right)^{\#\{-+\}}$$

Proof: Task 3

$$\leq \left(\frac{\sigma_m^4}{n\sigma_{nm}^4}\right)^{g_1} \left(\frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2}\right)^{g_2} \cdot n\sqrt{nm\sigma_{nm}^4 + k}^{\,k-k'+l-\#\{-+\}}$$

$$\times \left(n\sigma_n^2 + k\right)^{\#\{-+\}}$$

A!

Proof: Task 3

$$\leq \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\,\sigma_{nm}^2}\right)^g \cdot n\sqrt{nm\sigma_{nm}^4 + k}^{\,k-k'+l-\#\{-+\}}$$

$$\times \left(n\sigma_n^2 + k\right)^{\#\{-+\}}$$

Proof: Task 3

}

Task 3:

In contrary to task 2, the bound is smaller for larger genus

Proof: Task 4

$$\sum_{\#\{-+\},l,g} \text{Task1} \cdot \text{Task2} \cdot \text{Task3}$$

Proof: Task 4

$$= \sum_{\#\{-+\}} \sum_{g} \sum_{l} \binom{k}{\#\{-+\}} 2^{k-\#\{-+\}} \cdot \binom{2k'}{2l} \left(\frac{8e^4k^5}{g^3}\right)^g (2g)^{2k'-2l}$$

$$\times\, n \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm\sigma_{nm}^2}}\right)^g \left(\sqrt{nm\sigma_{nm}^4 + k}\right)^{k-k'+l-\#\{-+\}} \left(n\sigma_n^2 + k\right)^{\#\{-+\}}$$

A!

Proof: Task 4

$$= \sum_{\#\{-+\}} \sum_{g} \sum_{l} \binom{k}{\#\{-\ +\}} 2^{k-\#\{-+\}} \cdot \binom{2k'}{2l} \left(\frac{8e^4k^5}{g^3}\right)^{g} (2g)^{2k'-2l}$$

$$\times\, n \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2}\right)^{g} \left(\sqrt{nm\sigma_{nm}^4 + k}\right)^{k-k'+l-\#\{-+\}} \left(n\sigma_n^2 + k\right)^{\#\{-+\}}$$

Proof: Task 4

$$= n \sum_{\#\{-+\}} \binom{k}{\#\{-\ +\}} 2^{k-\#\{-+\}} \left(\frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}} \right)^{\#\{-+\}} \sum_g \left(\frac{8e^4k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^g \left(\sqrt{nm\sigma_{nm}^4 + k} \right)^{k-k'} \sum_l \binom{2k'}{2l} (2g)^{2k'-2l} \left(\sqrt{nm\sigma_{nm}^4 + k} \right)^l$$

Proof: Task 4

$$= n \sum_{\#\{-+\}} \binom{k}{\#\{-\ +\}} 2^{k-\#\{-+\}} \left(\frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}} \right)^{\#\{-+\}} \sum_g \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^g \left(nm\sigma_{nm}^4 + k\right)^{\frac{2k-2k'}{4}} \sum_l \binom{2k'}{2l} (2g)^{2k'-2l} \left(nm\sigma_{nm}^4 + k\right)^{\frac{2l}{4}}$$

Proof: Task 4

$$\leq n \sum_{\#\{-+\}} \binom{k}{\#\{-\ +\}} 2^{k-\#\{-+\}} \left(\frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}} \right)^{\#\{-+\}} \sum_g \left(\frac{8e^4k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm\sigma_{nm}^2}} \right) \right)^g (nm\sigma_{nm}^4 + k)^{\frac{2k-2k'}{4}} \left((nm\sigma_{nm}^4 + k)^{\frac{1}{4}} + 2g \right)^{2k'}$$

Proof: Task 4

$$= n \sum_{\#\{-+\}} \binom{k}{\#\{-\,+\}} 2^{k-\#\{-+\}} \left(\frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}} \right)^{\#\{-+\}} \sum_g \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm\sigma_{nm}^2}} \right) \right)^g (nm\sigma_{nm}^4 + k)^{\frac{2k}{4}} \left(\frac{(nm\sigma_{nm}^4 + k)^{\frac{1}{4}} + 2g}{(nm\sigma_{nm}^4 + k)^{\frac{1}{4}}} \right)^{2k'}$$

Proof: Task 4

$$\leq n \sum_{\#\{-+\}} \binom{k}{\#\{-\,+\}} 2^{k-\#\{-+\}} \left(\frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}} \right)^{\#\{-+\}} \sum_g \left(\frac{8e^4k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm\sigma_{nm}^2}} \right) \right)^g (nm\sigma_{nm}^4 + k)^{\frac{2k}{4}} \left(\frac{(nm\sigma_{nm}^4 + k)^{\frac{1}{4}} + 2g}{(nm\sigma_{nm}^4 + k)^{\frac{1}{4}}} \right)^{2k}$$

Proof: Task 4

$$\leq n \sum_{\#\{-+\}} \binom{k}{\#\{-\,+\}} 2^{k-\#\{-+\}} \left(\frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}}\right)^{\#\{-+\}} \sum_g \left(\frac{8e^4k^5}{g^3}\left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2}\right)\right)^g \left((nm\sigma_{nm}^4 + k)^{\frac{1}{4}} + 2g\right)^{2k}$$

Proof: Task 4

$$\leq n\left(2+\frac{n\sigma_n^2+k}{\sqrt{nm\sigma_{nm}^4+k}}\right)^k$$

$$\times\sum_g\left(\frac{8e^4k^5}{g^3}\left(\frac{\sigma_m^4}{n\sigma_{nm}^4}\vee\frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2}\right)\right)^g\left((nm\sigma_{nm}^4+k)^{\frac{1}{4}}+2g\right)^{2k}$$

Proof: Task 4

$$\leq n \sqrt{2 + \frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}}}^{2k}$$

$$\times \sum_g \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm\sigma_{nm}^2}} \right) \right)^g \left((nm\sigma_{nm}^4 + k)^{\frac{1}{4}} + 2g \right)^{2k}$$

Proof: Task 4

$$\leq n \sum_{g} \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\,\sigma_{nm}^2} \right) \right)^g$$

$$\times \sqrt{2 + \frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}}}^{\,2k} \left((nm\sigma_{nm}^4 + k)^{\frac{1}{4}} + 2g \right)^{2k}$$

Proof: Task 4

$$\leq n \sum_{g} \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^{g}$$

$$\times \left((nm\sigma_{nm}^4 + k)^{\frac{1}{4}} \sqrt{2 + \frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}}} + 2g \sqrt{2 + \frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}}} \right)^{2k}$$

A!

Proof: Task 4

$$\leq n \sum_g \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^g$$

$$\times \left((nm\sigma_{nm}^4 + k)^{\frac{1}{4}} \sqrt{2 + \frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}}} + 2g\sqrt{2} + 1 \right)^{2k}$$

Proof: Task 4

$$\leq n \sum_g \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^g$$

$$\times \left((nm\sigma_{nm}^4 + k)^{\frac{1}{4}} \sqrt{2 + \frac{n\sigma_n^2 + k}{\sqrt{nm\sigma_{nm}^4 + k}}} + 4g \right)^{2k}$$

Proof: Task 4

$$\leq n \sum_{g} \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^g$$

$$\times \left(\sqrt{2\sqrt{nm\sigma_{nm}^4 + k} + n\sigma_n^2 + k} + 4g \right)^{2k}$$

Proof: Task 4

$$\leq n \sum_{g} \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^g$$

$$\times \left(\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k} + 4g \right)^{2k}$$

Proof: Task 4

$$\leq n \sum_{g} \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^g$$

$$\times \left(\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k + 4g} \right)^{2k}$$

A!

Proof: Task 4

$$\leq nk \sup_{g} \left(\frac{8e^4 k^5}{g^3} \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\,\sigma_{nm}^2} \right) \right)^g$$

$$\times \left(\sqrt{2\sqrt{nm}\,\sigma_{nm}^2 + n\sigma_n^2 + 3k} + 4g \right)^{2k}$$

Proof: Task 4

$$= nk \sup_g \left(\frac{1}{g} \left(8e^4 k^5 \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^{\frac{1}{3}} \right)^{3g}$$

$$\times \left(\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k} + 4g \right)^{2k}$$

A!

Proof: Task 4

$$= nk \sup_g \left(\frac{e}{3g} \frac{3}{e} \left(8e^4 k^5 \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^{\frac{1}{3}} \right)^{3g}$$

$$\times \left(\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k} + 4g \right)^{2k}$$

A!

Proof: Task 4

$$= nk \sup_g \left(\frac{e}{3g} \frac{3}{e} \left(8e^4 k^5 \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^{\frac{1}{3}} \right)^{3g}$$

$$\times \left(\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k} + 4g \right)^{2k}$$

$$a = \frac{3}{e} \left(8e^4 k^5 \left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2} \right) \right)^{\frac{1}{3}}$$

Proof: Task 4

$$= nk \sup_{g} \left(\frac{ea}{3g}\right)^{3g}$$

$$\times \left(\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k} + 4g\right)^{2k}$$

$$a = \frac{3}{e}\left(8e^4k^5\left(\frac{\sigma_m^4}{n\sigma_{nm}^4} \vee \frac{\sigma_{\max}^2}{\sqrt{nm}\sigma_{nm}^2}\right)\right)^{\frac{1}{3}}$$

Proof: Task 4

$$= nk\left(\frac{4}{3}\right)^{2k}\ \sup_{g}\left(\frac{ea}{3g}\right)^{3g} \times \left(\frac{3}{4}\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k} + 3g\right)^{2k}$$

$$b = \frac{3}{4}\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k}$$

Proof: Task 4

$$= nk\left(\frac{4}{3}\right)^{2k} \sup_{g}\left(\frac{ea}{3g}\right)^{3g} \times (b+3g)^{2k}$$

$$b = \frac{3}{4}\sqrt{2\sqrt{nm}\sigma_{nm}^2 + n\sigma_n^2 + 3k}$$

Proof: Task 4

$$= nk\left(\frac{4}{3}\right)^{2k} e^{a}\left(b+\frac{2k}{W\left(\frac{2k}{a}\right)}\right)^{2k}$$

Proof: Task 4

$$= nke^{a}\left(\sqrt{2\sqrt{nm}\sigma_{nm}^{2} + n\sigma_{n}^{2} + 3k} + \frac{8k}{3W\left(\frac{2k}{a}\right)}\right)^{2k}$$