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Fully dynamic risk measures and BSDEs

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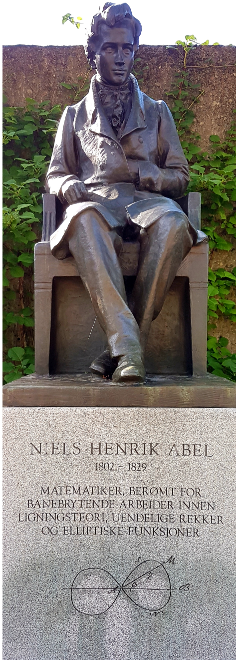

Risk and Ambiguity vs Uncertainty

Uncertainty[1] is the general lack of sureness. Uncertainty is a status that can be better (partially) understood thanks to the analysis of the chaotic situation.

Risk is defined as a quantifiable uncertainty. Typically the potential outcomes can be described in a set of scenarios in which a probability measure is given. Risk refers to a possible negative departure from a reference acceptance level.

So, risk does not (just) mean variability. The study of risks reduces the uncertainty into risk management, thanks to risk measures.

Ambiguity is the context in which the scenarios are known, but no referent probability is possible to be identified precisely. In this case, risk analysis needs to be coupled with some form of robustness.

[1] Knight (1921), Ellsberg (1961),... Riedel (2019,2021).

PART I

- Risk measures from static point of view, properties
- Dynamic risk measurement
- BSDEs of different types of interest
- Relation BSDEs and dynamic risk measurement
- Horizons and time: risk and consistency

Past, present, future and information

We are interested in risks associated to phenomena evolving in time.

So, in time dynamics,

- the future scenarios are represented by $(\Omega, \mathcal{F}, P)$.
- The information flow builds up in time $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $s \leq t$.
- For all t, $\mathcal{X}_t$ represents $\mathcal{F}_t$-random variables.

1. Static risk measures

A static risk measure is a mapping

$$\rho : \mathcal{X}_T \longrightarrow \mathcal{X}_0 \quad (= \mathbb{R})$$

with some properties:

1. monotone: if $X \leq Y$, then $\rho(X) \geq \rho(Y)$
2. translation invariant/additive: if $m \in \mathcal{X}_0$, then $\rho(X + m) = \rho(X) - m$
3. positive homogeneous: for $\lambda > 0$, then $\rho(\lambda X) = \lambda \rho(X)$
4. normalized: $\rho(0) = 0$
5. sub-additive: $\rho(X + Y) \leq \rho(X) + \rho(Y)$
6. **convex**: for $\lambda \in [0, 1]$, then $\rho\Big(\lambda X + (1 - \lambda) Y\Big) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y)$
7. law invariant: if $\mathcal{L}(X) = \mathcal{L}(Y)$, then $\rho(X) = \rho(Y)$

- Normalization can be included in positive homogeneous, taking $\lambda \geq 0$.
- A risk measure with properties 1-5 is called coherent.
- A **monetary evaluation** of an admissible risk:

$$\rho(X) = inf\{m: \ m + X \in \mathcal{A}\}.$$

- **Diversification** is realized by sub-additivity, i.e. a portfolio of a pool of different positions bears less risk then the sum of risks of the different positions separately.
- **Diversification** is also identified by **convexity**. Indeed, a convex combination is a mixture of two positions. The risk of the mixture cannot be worse than the average of the two individual risks.
  Result. Sub-additivity and positive homogeneity imply convexity.
  Indeed,

$$\rho\big(\lambda X + (1-\lambda)Y\big) \overset{\text{subadd.}}{\leq} \rho(\lambda X) + \rho((1-\lambda)Y) \overset{\text{pos. hom.}}{=} \lambda\rho(X) + (1-\lambda)\rho(Y).$$

- A risk measure with properties 1-2, 6 is called convex risk measure
- Dropping positive homogeneity does not force linear scaling. This is adequate when modeling the risk associated to large positions, or in the case of illiquidity.
- The role of **normalization** is possibly correct in the very context of financial markets: Zero position equals to zero risk. However, in many contexts the strategy of "do nothing" can bear risk. Think about health, environment, but also the non-playing even in the financial market for guaranteeing capital. So normalization is contextual.
- Law invariant relates the risk to the distribution of the position, ignoring which scenario produces the critical risk. So non-law invariant risk measures are state dependent. For example there can be weights that are particularly critical then the risk measure is weighted over these (as a scenario-based evaluation)
- Convex (additive, law invariant) risk measures have a **convex dual representation** (Fenchel-Legendre type), which opens connection with convex analysis. For $\mathcal{X}_T = L^\infty(\mathcal{F}_T)$

$$\rho(X) = \sup_{Q \in \mathcal{M}_1} \left(\mathbb{E}_Q[-X] \; - \; \alpha(Q) \right)$$

where $\mathcal{M}_1$ are the probabilities Q: $Q << P$ and $\alpha : \mathcal{M}_1 \to (-\infty, +\infty]$ is a penalty function

Some notable cases of large use[2]

- **Value-at-Risk** $\alpha \in (0,1)$ of a loss X is defined as the (lower) quantile of the distribution of X:

$$VaR_\alpha(X) := \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \alpha\}.$$

VaR is monotone, translation invariant, normalized, positive homogeneous, but not sub-additive in general (hence VaR penalizes diversification). It is law invariant.

Example. Consider two iid random variables X_1, X_2 and their sum S:

$$X_i = \begin{cases} 100, & P\{X_i = 100\} = 4\% \\ 0, & P\{X_i = 0\} = 96\% \end{cases} \qquad S = X_1 + X_2 = \begin{cases} 200, & P\{S = 200\} = 0.16\% \\ 100, & P\{S = 100\} = 7.68\% \\ 0, & P\{S = 0\} = 92.16\% \end{cases}$$

Then

$$VaR_{95\%}(S) = 100 > VaR_{95\%}(X_1) + VaR_{95\%}(X_2) = 0 + 0 = 0$$

[2] Banking - Basil I, II, III regulatory framework for risk management with credit, market, and operational risk.
Insurance and reinsurance - Solvency I, II prudential regime framework

- **Expected Shortfall (ES) or conditional VaR (CVaR)** $\alpha \in (0, 1)$:

$$CVaR_\alpha(X) = ES_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(X) du$$

This is a coherent risk measure, also law invariant.

ES does not refer just to the threshold, but to the whole tail. So $VaR_\alpha(X)$ measures the entry into the critical severity of losses, while $ES_\alpha(X)$ evaluates the average severity of losses within the critical zone.

From a practical perspective often regulators appreciate ES_α as it is sensitive to tail event measuring how serious would be the situation in a critical set-up. In Basel framework, financially ES_α has basically replaced the initial VaR_α in view of the best capturing of diversification

(A Success Story for Mathematics - Artzner, Delbaen, Eber, Heath, Föllmer, Schied, Acerbi, Tasche, Kusuoka, Embrechts, ...)

- **Entropic risk measure $\theta > 0$**:

$$\rho^\theta(X) = \frac{1}{\theta}\log E\left[e^{-\theta X}\right] = \sup_{Q\in\mathcal{M}_1}\left\{E_Q[-X] - \frac{1}{\theta}H(Q|P)\right\}$$

relative entropy $H(Q|P) := E\left[\frac{dQ}{dP} log \frac{dQ}{dP}\right]$.

ρ^θ is not coherent because it is not positive homogeneous. Indeed, for $\lambda > 0$,

$$\rho^\theta(\lambda X) = \frac{1}{\theta}\log \mathbb{E}[e^{-\theta\lambda X}] \quad \neq \quad \lambda\rho^\theta(X) = \frac{\lambda}{\theta}\log \mathbb{E}[e^{-\theta X}].$$

However, the measure is law invariant and convex. About convexity:

$$\begin{aligned}\rho\big(\lambda X + (1-\lambda)Y\big) &= \frac{1}{\theta}\log \mathbb{E}\big[e^{-\theta(\lambda X + (1-\lambda)Y)}\big] = \frac{1}{\theta}\log \mathbb{E}\big[e^{-\theta\lambda X}\, e^{-\theta(1-\lambda)Y}\big] \\ &\leq \frac{1}{\theta}\log\Big(\big(\mathbb{E}[e^{-\theta X}]\big)^{\lambda}\big(\mathbb{E}[e^{-\theta Y}]\big)^{1-\lambda}\Big) \\ &= \frac{1}{\theta}\Big(\lambda\log \mathbb{E}[e^{-\theta X}] + (1-\lambda)\log \mathbb{E}[e^{-\theta Y}]\Big) = \lambda\rho(X) + (1-\lambda)\rho(Y).\end{aligned}$$

2. Dynamic risk measures: past, present, and future

Why dynamic risk measuring?

- Time-evolving phenomena carry flows of information, that has to be captured as this effects risk
- Risk evaluation effects decision making, which needs to be updated for better performance and less costly re-adjusting of strategies
- regulators require multi-period assessments for strategies that would require fine-tuning, such as liquidation

In time dynamics,

- the future scenarios are represented by $(\Omega, \mathcal{F}, P)$.
- The information flow builds up in time $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $s \leq t$.
- For all t, $\mathcal{X}_t$ represents $\mathcal{F}_t$-random variables. We mostly work with $\mathcal{X}_t = L^p(\mathcal{F}_t)$ for some $p \in [1, \infty]$ and we take $\mathcal{X}_t = L^2(\mathcal{F}_t)$ for the standard BSDE framework.

A dynamic risk measure is a family of individual risk measures $(\rho_t)_{0 \leq t \leq T}$:

$$\rho_t : \mathcal{X}_T \longrightarrow \mathcal{X}_t$$

We assume:

- monotonicity
- additivity/translation invariance
- convexity

Dynamic risk measures and BSDEs - Gaussian case

Considering the future uncertainty to be of Gaussian nature, Brownian motion can be taken as noise with the associated information flow.

Characterisation of rm in terms of BSDEs.[3] Dynamic risk measures are associated to BSDEs (= Backward Stochastic Differential Equations):

$$Y_t = X + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s$$

The process $(Y_t)_t$ in the solution $(Y_t, Z_t)_{t\in[0,T]}$ is regarded as an operator depending on the **driver** g and evaluated at $X \in L^2(\mathcal{F}_T)$, which turns out to represent the nonlinear expectations

$$\mathcal{E}^g(X|\mathcal{F}_t) = Y_t, \qquad X \in L^2(\mathcal{F}_T),\ t \in [0, T].$$

Depending on the properties of g, we have $\rho_t(X) = \mathcal{E}^g(-X|\mathcal{F}_t)$.

[3] Peng (1997, 2003), Frittelli, Rosazza Gianin (2002, 2004), Rosazza Gianin (2006), ...

3. About BSDEs in a Gaussian framework

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a d-dimensional Brownian motion $W = (W_t)_{0 \leq t \leq T}$.
- Augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ generated by W.
- Terminal time $T > 0$ fixed.
- Terminal condition $\xi : \Omega \to \mathbb{R}$, $\mathcal{F}_T$-measurable.
- Generator (driver)
$$g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$$
progressively measurable.

Definition of a BSDE

The solution to a BSDE is given by the couple $(Y_t, Z_t)_{0 \leq t \leq T}$ such that

$$
\begin{aligned}
& Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad 0 \leq t \leq T, \\
& Y \in \mathcal{S}^2, \quad Z \in \mathcal{H}^2,
\end{aligned}
$$

where

- $\mathcal{S}^2 = \{Y$ adapted, continuous : $\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^2] < \infty\}$,
- $\mathcal{H}^2 = \{Z$ prog. measurable : $\mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < \infty\}$.

2.1 Existence and unicity of a solution - Lipschitz case

Standard assumptions (Pardoux–Peng):

- Terminal condition: $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.
- Integrability of $g(\cdot, 0, 0)$:

$$\mathbb{E}\left[\int_0^T |g(t,0,0)|^2 dt\right] < \infty.$$

- Lipschitz driver: there exists $L > 0$ s.t. for all $(y, z), (y', z')$,

$$|g(t,y,z) - g(t,y',z')| \leq L\big(|y-y'| + |z-z'|\big), \quad P\text{-a.s. and } t\text{-a.e. } .$$

Theorem (Pardoux–Peng, 1990).

Under the standard assumptions, there exists a unique pair $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$ solving

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s.$$

Idea of the proof:

- Define a mapping on $\mathcal{S}^2 \times \mathcal{H}^2$ via the BSDE.
- Use Itô isometry and Gronwall inequality to show it is a contraction.
- Apply Banach fixed point theorem to obtain existence and uniqueness.

Comparison Theorem

In the framework of the standard assumptions

- Two BSDEs with data (ξ^1, g^1) and (ξ^2, g^2).
- and solutions (Y^1, Z^1) and (Y^2, Z^2).

Assumptions:

- $\xi^1 \le \xi^2$ a.s.
- For all t, for all $y \in \mathbb{R}$, $z \in \mathbb{R}^d$,

$$g^1(t, y, z) \le g^2(t, y, z) \quad \text{a.s.}$$

Conclusion (Comparison):

$$Y_t^1 \le Y_t^2, \quad 0 \le t \le T, \quad \text{a.s.}$$

Convexity of Y in the Terminal Condition (Setup)

Let us mark the terminal condition impact explicitly in the notation:

$$Y_t^\xi = \xi + \int_t^T g(s, Y_s^\xi, Z_s^\xi)\, ds - \int_t^T Z_s^\xi\, dW_s, \quad 0 \le t \le T.$$

We assume that there is a solution and that the driver is convex: For all (t, ω) and $\lambda \in [0, 1]$,

$$g(t, \lambda y_1 + (1-\lambda) y_2, \lambda z_1 + (1-\lambda) z_2) \le \lambda g(t, y_1, z_1) + (1-\lambda) g(t, y_2, z_2), \quad \forall (y_1, z_1), (y_2, z_2).$$

Then, for all t, the solution process Y_t^ξ is convex in ξ.

Proof (sketch)

Let $\xi^1, \xi^2 \in L^2(\mathcal{F}_T)$ and $\lambda \in [0,1]$.

Goal: For each t, we have to show convexity of the mapping $\xi \mapsto Y_t^{\xi}$ i.e.

$$Y_t^{\lambda\xi^1+(1-\lambda)\xi^2} \leq \lambda Y_t^{\xi^1} + (1-\lambda) Y_t^{\xi^2}, \quad \text{a.s.}$$

Step 1: Define the convex combination of the two solutions

$$\bar{Y}_t := \lambda Y_t^{\xi^1} + (1-\lambda) Y_t^{\xi^2}, \qquad \bar{Z}_t := \lambda Z_t^{\xi^1} + (1-\lambda) Z_t^{\xi^2}.$$

Using the two BSDEs, we find the dynamics of $\bar{Y}_t$. This is associated to the convex combinations of the terminal conditions and of the drivers

$$\bar{Y}_t = \left(\lambda\xi^1 + (1-\lambda)\xi^2\right) + \int_t^T \left[\lambda g(s, Y_s^{\xi^1}, Z_s^{\xi^1}) + (1-\lambda) g(s, Y_s^{\xi^2}, Z_s^{\xi^2})\right] ds - \int_t^T \bar{Z}_s \, dW_s. \tag{1}$$

Step 2: By the convexity of the driver g:

$$\lambda g(s, Y_s^{\xi^1}, Z_s^{\xi^1}) + (1-\lambda) g(s, Y_s^{\xi^2}, Z_s^{\xi^2}) \geq g\big(s, \bar{Y}_s, \bar{Z}_s\big).$$

We have that

$$\bar{Y}_t \geq \lambda \xi^1 + (1-\lambda)\xi^2 + \int_t^T g(s, \bar{Y}_s, \bar{Z}_s)\, ds - \int_t^T \bar{Z}_s\, dW_s.$$

Step 3: Now we recognize the dynamics on the right-hand side above as the original BSDE with terminal condition given by the convex combination with the solution of the BSDE with terminal condition $\lambda \xi^1 + (1-\lambda)\xi^2$:

$$Y_t = \lambda\Big(\xi^1 + (1-\lambda)\xi^2\Big) + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s. \tag{2}$$

The solution is denoted $(Y^{\lambda\xi^1+(1-\lambda)\xi^2}, Z^{\lambda\xi^1+(1-\lambda)\xi^2})$.

Step 4: The Comparison Theorem between (1) and (2) allows to conclude that

$$Y_t^{\lambda\xi^1+(1-\lambda)\xi^2} \leq \bar{Y}_t = \lambda Y_t^{\xi^1} + (1-\lambda) Y_t^{\xi^2}, \quad \forall t, \text{ a.s.}$$

Examples of drivers both convex and Lipschitz

1. **Affine + norm driver**
   - Let $\alpha_t, \beta_t, \gamma_t$ be bounded progressively measurable.
   - $g(t, y, z) \; := \; \alpha_t \, y \; + \; \beta_t \, |z| \; + \; \gamma_t.$
2. **Positive/negative parts driver**
   - Let $\alpha_t, \beta_t, \gamma_t$ be bounded progressively measurable.
   - $g(t, y, z) \; := \; \alpha_t \, y^+ \; + \; \beta_t \, y^- \; + \; \gamma_t \, |z|.$
3. **Hamiltonian / control-type driver**
   - Let A be compact, and $b, \sigma, c : [0, T] \times A \to \mathbb{R}$ bounded.
   - $g(t, y, z) \; := \; \sup_{a \in A} \Big(b(t, a) \, y \; + \; \sigma(t, a) \cdot z \; + \; c(t, a) \Big).$

2.2 Existence and Uniqueness - Quadratic case

The quadratic case

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s, \qquad t \in [0, T],$$

with generator satisfying

$$|g(t, y, z)| \;\le\; C\big(1 + |y| + |z|^2\big).$$

Key point: the theory is more delicate than for Lipschitz drivers.

Assumptions in the bounded terminal condition (Kobylanski 2000)

- ξ is bounded: $|\xi| \leq M$ a.s.
- g is continuous in (y, z).
- **Quadratic growth in z:** For some $C \geq 0$ and $\alpha \geq 0$

$$|g(t, y, z)| \leq C\big(1 + |y|\big) + \frac{\alpha}{2} |z|^2.$$

- **Monotonic/Lipschitz in y:** for some $\mu \in \mathbb{R}$,

$$(y - y')\big(g(t, y, z) - g(t, y', z)\big) \leq \mu |y - y'|^2.$$

Result.

There exists a solution (Y, Z) with Y bounded and $\int_0^\cdot Z_s \, dB_s$ a BMO-martingale.

The solution is **unique** in the class of bounded solutions (or, more generally, in an appropriate BMO class).

Role of BMO and Structural Properties

BMO = Bounded Mean Oscillation

Definition BMO (Bounded Mean Oscillation)-Martingales are martingales such that

$$\|M\|^2_{\mathrm{BMO}} := \sup_{\tau} \left\| \mathbb{E}\left[|M_T - M_\tau|^2 \,\middle|\, \mathcal{F}_\tau \right] \right\|_{L^\infty(\mathbb{P})} < \infty,$$

where the sup is taken over all stopping times.

For stochastic integrals with respect to Brownian motion one often uses the equivalent formulation:

Result Let B be a d-dimensional Brownian motion and let $M_t := \int_0^t Z_s \, dB_s$, $t \in [0, T]$. Then BMO property is expressed as

$$\sup_{\tau} \left\| \mathbb{E}\left[\int_\tau^T |Z_s|^2 \, ds \,\middle|\, \mathcal{F}_\tau \right] \right\|_{L^\infty(\mathbb{P})} < \infty$$

- A BMO-martingale is one whose "conditional future variability" is uniformly controlled, no matter at what random time you look at it.
- In the BSDE context, for $M_t = \int_0^t Z_s \, dB_s$, being BMO is equivalent to $\sup_\tau \mathbb{E}\big[\int_\tau^T |Z_s|^2 ds \mid \mathcal{F}_\tau\big] < \infty$, which guarantees, for example, that certain stochastic exponentials are true martingales.

  Condition that is crucial for Girsanov transforms where the transform is

$$\mathcal{E}(\lambda M)_T = \exp\Big(\lambda \int_0^T Z_s \, dB_s - \frac{\lambda^2}{2} \int_0^T |Z_s|^2 \, ds\Big).$$

- In the quadratic case, uniqueness holds in general in some class for (Y, Z) and not for the family of adapted processes.

A family of quadratic BSDEs

Theorem (Bahlali, Eddabbi, Ouknine 2017, Rosazza Gianin, DN 2026.)

For $\xi \in L^2(\mathcal{F}_T)$ and

$$g(t, y, z) = a_t + f(y)|z|^2 \tag{3}$$

with $a \in L^1([0, T])$ and $f \in L^1(\mathbb{R})$, the corresponding BSDE has a unique solution in the square integrable adapted processes $\mathbb{H}^2_{[0,T]}(\mathbb{R}) \times \mathbb{H}^2_{[0,T]}(\mathbb{R}^d)$. [4] Moreover, for

non-negative a and f, the Y-component of the unique solution is non-negative.

[4] $H^2_{[0,T]}(\mathbb{R}^d) := \mathbb{H}^2_{[a,b]}(\mathbb{R}^k) := \left\{ \text{adapted } \mathbb{R}^k\text{-valued processes } (\eta_t)_{t\in[a,b]} : E\left[\int_a^b |\eta_t|^2\, dt\right] < \infty \right\}$.

Quadratic BSDEs and entropic dynamic risk measure

Theorem

Let $\theta > 0$ and $\xi \in L^2(\mathcal{F}_T)$. Then:

1. The BSDE

$$Y_t = \xi + \int_t^T \frac{\gamma}{2} |Z_s|^2 \, ds - \int_t^T Z_s \, dB_s, \qquad t \in [0, T],$$

   admits a unique solution (Y, Z) in the square integrable adapted processes.

2. This solution is given by

$$Y_t = -\frac{1}{\theta} \log \mathbb{E}\left[e^{-\theta\xi} \mid \mathcal{F}_t\right] = \rho_t(\xi), \qquad t \in [0, T].$$

Proof.

The solution (Y, Z) exists in view of the Theorem of Existence for drivers of type (3). Let $\mathcal{Z}$ be an adapted process in $L^2(\Omega \times [0, T]$. Define

$$V_t := e^{-\theta X} - \int_t^T \mathcal{Z}_s dB_s$$

Take the Ito formula for

$$\begin{aligned} d\Big(-\frac{1}{\theta}\ln V_t\Big) &= -\frac{1}{\theta}\Big(-\frac{1}{V_t}dV_t - \frac{1}{2}\frac{1}{V_t^2}d\langle V\rangle_t\Big) \\ &= -\frac{\mathcal{Z}_t}{\theta V_t}\,dB_t + \frac{\theta}{2}\,\frac{\mathcal{Z}_t^2}{\theta^2 V_t^2}\,dt \\ &= -\tilde{Z}_t dB_t + \frac{\theta}{2}\tilde{Z}_t^2 dt \qquad \text{with } \widetilde{Z}_t := \frac{\mathcal{Z}_t}{\theta V_t}. \end{aligned}$$

So the process $\widetilde{\mathcal{Y}}_t := -\frac{1}{\theta}\ln V_t$ in couple $(\mathcal{Y}, \mathcal{Z})$ is a solution to the quadratic BSDE.

In view of the uniqueness of the solution, we have that

$$Y_t = \mathcal{Y}_t = -\frac{1}{\theta} \ln V_t$$

is the unique solution to the quadratic BSDE. This also shows that V_t is $\mathcal{F}_t$-measurable. Hence,

$$V_t = E\Big[e^{-\theta X} + \int_t^T \mathcal{Z}_s dB_s | \mathcal{F}_t\Big] = E\Big[e^{-\theta X} | \mathcal{F}_t\Big]$$

. Altogether, the above brings to the representation

$$Y_t = -\frac{1}{\theta} \ln V_t = -\frac{1}{\theta} \ln E\Big[e^{-\theta X} | \mathcal{F}_t\Big]$$

.

3. Dynamic Risk Measures via BSDEs[5] [6]

Definition Let (Y_t, Z_t) solve

$$Y_t = X + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s.$$

Define the *dynamic risk measure* by $\rho_t(X) := -Y_t^X$.

Properties for suitable driver g:

- convexity of g implies convexity of the measure
- monotonicity is directly implied by the Comparison theorem
  $X_1 \leq X_2 \Rightarrow \rho_t(X_1) \geq \rho_t(X_2)$.
- If $g(t, 0, 0) = 0$, then normalization is guaranteed, immediate by substitution.

[5] Rosazza Gianin, 2006

[6] The future may not be Gaussian, other family of noises can be considered. See, e.g. Royer (2006), Quenez, Sulem (2013), Laeven, Stadje(2014, DiNunno, Sjursen (2014), Sulem, Øksendal (2019),...

Examples of NON-Normalized Risk Measures

a) Consider now the driver $g(t,z) = bz + a$ with $a, b \in \mathbb{R} \setminus \{0\}$. Then

$$\rho_s(X) = E_Q\left[-X \,|\, \mathcal{F}_s\right] + a(T-s),$$

where $E_P\left[\frac{dQ}{dP}\Big|\, \mathcal{F}_t\right] = \exp\left\{-\frac{1}{2}b^2 t + b \cdot B_t\right\}$.

For $a \neq 0$, $(\rho_s)_s$ is NOT normalized.

b) Entropic type risk measures (quadratic BSDEs). In the one-dimensional case, consider

$$Y_s = -X + \int_s^T \left[b\frac{Z_r^2}{2} + a\right] dr - \int_s^T Z_r dB_r$$

with positive a, b. Then we have

$$\rho_s(X) = \frac{1}{b} \ln\left(E_P\left[\exp(-bX) \,|\, \mathcal{F}_s\right]\right) + a(T-s).$$

Hence, $(\rho_{sT})_{s,T}$ is NOT normalized.

Properties for suitable driver g:

- Translation invariance if f has appropriate structure (e.g. g independent of y)[7].

  Proposition.

  ρ is translation invariant, i.e. for all $c \in \mathbb{R}$ and all ξ,

  $$\rho_t(\xi + c) = \rho_t(\xi) - c, \qquad \forall t \in [0, T].$$

  if and only if the driver g is independent of y.

[7] Barrieu and El Karoui (2009), Jiang (2008).

Proof.

1) Assume

$$g(t, y, z) = g(t, z) \quad \text{for all } (t, y, z).$$

(Y, Z) solves the BSDE with terminal condition ξ:

$$Y_t = \xi + \int_t^T g(s, Z_s)\, ds - \int_t^T Z_s\, dB_s.$$

Fix $c \in \mathbb{R}$ and define

$$\tilde{Y}_t := Y_t + c, \qquad \tilde{Z}_t := Z_t.$$

Then

$$\tilde{Y}_t = Y_t + c = (\xi + c) + \int_t^T g(s, \tilde{Z}_s)\, ds - \int_t^T \tilde{Z}_s\, dB_s.$$

Thus $(\tilde{Y}, \tilde{Z})$ solves the BSDE with terminal condition $\xi + c$. By uniqueness of the solution, $Y_t(\xi + c) = \tilde{Y}_t = Y_t(\xi) + c$. And the corresponding risk measure is translation invariant.

2) Assume the risk measure is translation invariant.

Fix ξ and let (Y^ξ, Z^ξ) solve

$$Y_t^\xi = \xi + \int_t^T g(s, Y_s^\xi, Z_s^\xi)\, ds - \int_t^T Z_s^\xi\, dB_s.$$

Fix a constant $c \in \mathbb{R}$ and define the shifted processes

$$\tilde{Y}_t := Y_t^\xi + c, \qquad \tilde{Z}_t := Z_t^\xi.$$

Moreover,

$$d\tilde{Y}_t = dY_t^\xi = -g(t, Y_t^\xi, Z_t^\xi)\, dt + Z_t^\xi\, dB_t = -g\big(t, \tilde{Y}_t - c, \tilde{Z}_t\big)\, dt + \tilde{Z}_t\, dB_t.$$

Thus $(\tilde{Y}, \tilde{Z})$ solves the BSDE

$$\tilde{Y}_t = \xi + c + \int_t^T \tilde{g}(s, \tilde{Y}_s, \tilde{Z}_s)\, ds - \int_t^T \tilde{Z}_s\, dB_s,$$

with *modified driver* $\tilde{g}(t, y, z) := g(t, y - c, z)$.

Furthermore, from translation invariance, we have

$$Y_t^{\xi+c} = Y_t^{\xi} + c = \tilde{Y}_t, \qquad \forall t \in [0, T].$$

With this we have:

- $(Y^{\xi+c}, Z^{\xi+c})$ solves the BSDE with driver g and terminal $\xi + c$;
- $(\tilde{Y}, \tilde{Z})$ solves the BSDE with driver $\tilde{g}$ and terminal $\xi + c$;
- and $Y^{\xi+c} \equiv \tilde{Y}$.

By uniqueness of the BSDE solution, we must also have

$$Z^{\xi+c} \equiv \tilde{Z},$$

and the two drivers must coincide

$$g(t, \tilde{Y}_t, \tilde{Z}_t) = \tilde{g}(t, \tilde{Y}_t, \tilde{Z}_t) = g(t, \tilde{Y}_t - c, \tilde{Z}_t), \quad \text{a.s. for all } t.$$

Thus, for this BSDE solution,

$$g\big(t, Y_t^{\xi} + c, Z_t^{\xi}\big) = g\big(t, Y_t^{\xi}, Z_t^{\xi}\big) \quad \text{for all } c \in \mathbb{R}.$$

The random pairs $\big(Y_t^{\xi}, Z_t^{\xi}\big)$ as ξ varies, can take arbitrary values in $\mathbb{R} \times \mathbb{R}^d$ (at least on sets of positive probability). Hence we can rewrite the above as:

$$g(t, y + c, z) = g(t, y, z), \quad \forall y, z, c, \text{ for a.e. } t.$$

This implies that g is constant in its y-argument:

$$g(t, y, z) = g(t, 0, z), \quad \text{i.e. } g \text{ does not depend on } y.$$

4. Dynamic risk evaluation, horizon, time consistency

Recall that dynamic risk evaluation is used substantially for being able to make decision over time and adjust them taking care of the incoming information.

However, we have to be careful

- to make coherent decisions without regret. Namely we need some form of time consistency [8]
- We have to be careful of the time horizon at stake: short, long, mixed?

  In particular, we have to be cautious not to introduce some Horizon Risk in the dynamic evaluation so that instead of doing good we mess up.[9]

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[8] Acciaio, Penner (2011), Bielecki, Cialenco, Pitera (2017), Bion-Nadal (2009, Bion-Nadal, DN (2020).)

[9] DN, Rosazza Gianin (2024, 2026)

PART I - Where were we?

- Risk measures from static point of view, properties
- Dynamic risk measurement
- BSDEs of different types of interest
- Relation BSDEs and dynamic risk measurement
- Horizons and time: risk and consistency

Dynamic risk evaluation, horizon, time consistency

Recall that dynamic risk evaluation is used substantially for being able to make decision over time and adjust them taking care of the incoming information.

However, we have to be careful

- to make coherent decisions without regret. Namely we need some form of time consistency [10]
- We have to be careful of the time horizon at stake: short, long, mixed?

  In particular, we have to be cautious not to introduce some Horizon Risk in the dynamic evaluation so that instead of doing good we mess up.[11]

[10] Acciaio, Penner (2011), Bielecki, Cialenco, Pitera (2017), Bion-Nadal (2009, Bion-Nadal, DN (2020).)

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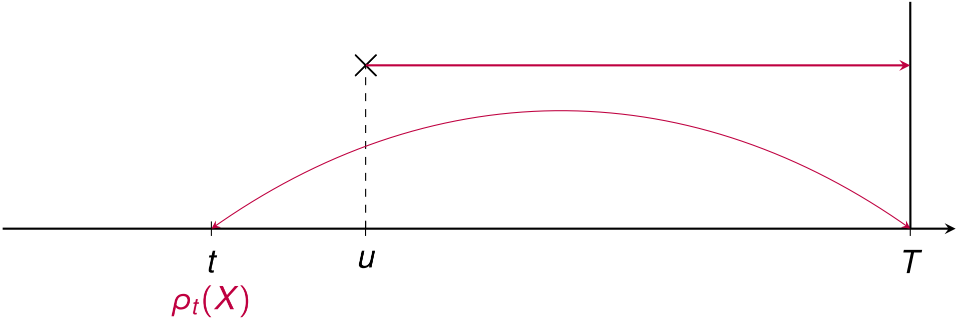
PART II

- Dynamic risk evaluation: the horizon and time consistency
- Fully-dynamic risk measures: restriction and normalization
- Time consistency definitions and h-longevity: interrelationships
- Fully-dynamic risk measures and BSDEs
- Outlook: Family of BSDEs, perhaps?
- Outlook: Even further with Volterra BSDEs, maybe?

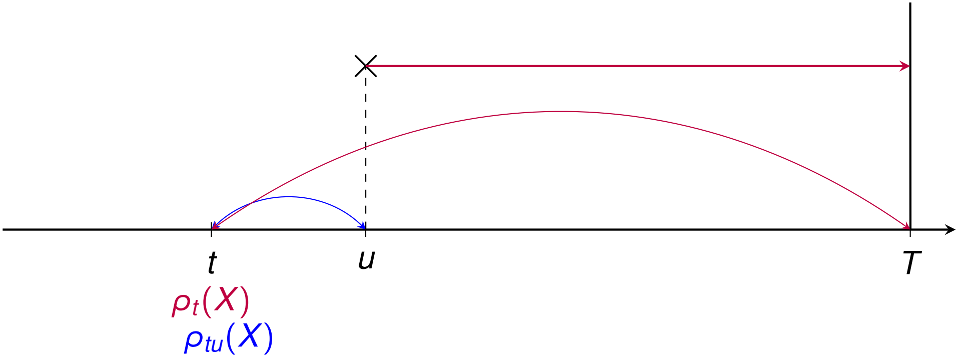
Dynamic evaluations and Horizon risk



- If $X \in \mathcal{X}_T$, we use the measure $\rho_t(X)$
- If $X \in \mathcal{X}_u$ with $u << T$, we use the same measure $\rho_t(X)$.

This is justified form information point of view as $\mathcal{F}_u \subseteq \mathcal{F}_T$.

What about other uncertainties that may appear in the long run?



Horizon risk emerges with the use of a wrong risk measure for the targeted horizon.

- We introduce fully dynamic risk measures
- We characterize the relationship with the ordinary dynamic risk measures
- We quantify horizon risk by the h-longevity

Time consistency - in what sense?

In general it all deals with preserving some concept of maintaining some rationale in what is considered risk at the different evaluation times.

- The intuition of something acceptable for decision making. In the normalized setting $\rho_t(0) = 0$, a position in the future is acceptable $\rho_t(X) \leq 0$, then it must be acceptable at $\rho_s(X) \leq 0$ for $s \leq t$.
- The intuition of comparison of risks. If two risky positions $\rho_t(X) \geq \rho_t(Y)$, then it should also be $\rho_s(X) \geq \rho_s(Y)$, for $s \leq t$.
- The intuition that one can measure risk in different operational steps. If we measure $\rho_t(X)$ and then we take $\rho_s(\rho_t(X)$ it should be the same as doing this in one round $\rho_s(X)$.

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- The intuition of comparison of risks. If two risky positions $\rho_t(X) \geq \rho_t(Y)$, then it should also be $\rho_s(X) \geq \rho_s(Y)$, for $s \leq t$.
- The intuition that one can measure risk in different operational steps. If we measure $\rho_t(X)$ and then we take $\rho_s(-\rho_t(X)$ it should be the same as doing this in one round $\rho_s(X)$.

So, in order to be able to fit both time consistency and track the horizon structures we introduce **fully-dynamic risk measures**.

1. Fully-dynamic risk measures

Fully-dynamic (convex) risk measure[12] is a *family* $(\rho_{tu})_{t,u}$ of risk measures :

$$\rho_{tu} : \mathcal{X}_u \longrightarrow \mathcal{X}_t$$

In many applications, we consider each of the risk meas. satisfying

- monotonicity, convexity
- $\mathcal{F}_s$-translation invariance or cash additivity , i.e. for $X \in \mathcal{X}_u$,

$$\rho_{tu}(X + m) = \rho_{tu}(X) - m, \text{ for all } m \in \mathcal{X}_t$$

Often we use $\mathcal{X}_u = L_p(\mathcal{F}_u)$, for $p \in [1, \infty]$. Then we also consider

- If $p = \infty$, also *continuity from below*, i.e. for any $X \in L_\infty(\mathcal{F}_u)$ and $(X_n)_n$ such that $X_n \uparrow X$ P-a.s., then $\rho_{tu}(X_n) \downarrow \rho_{tu}(X)$ P-a.s. [13]

[12] Bion-Nadal, DN (2020)

[13] If $p < \infty$, ρ_{st} is always *continuous from below* both in the L_p-convergence and P-a.s. See e.g. Biagini, Frittelli (2009), Filipovic, Svinland (2008)

- We observe that the risk measure ρ_{st} satisfies *weak $\mathcal{F}_s$-homogeneity*, i.e.

$$1_A \rho_{tu}(X) = 1_A \rho_{tu}(1_A X), \qquad X \in L_p(\mathcal{F}_t), \quad A \in \mathcal{F}_t.$$

- We do not assume apriori that the risk measures are normalized.

We recall that **normalization** is

$$\rho_{tu}(0) = 0$$

- We do not assume that the risk measures have the restriction property.

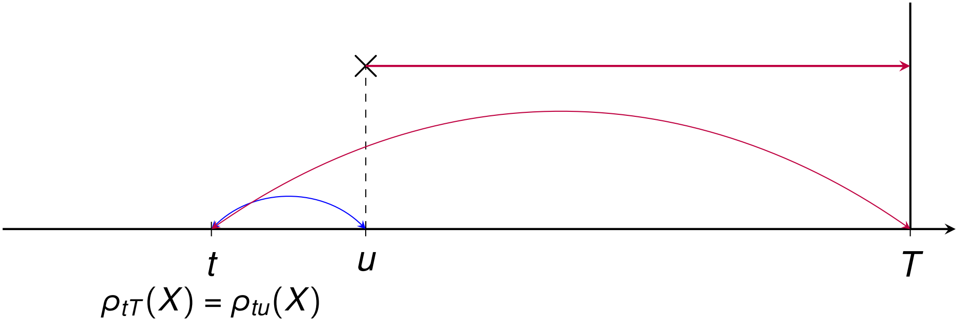
We define the **restriction property** as

$$\rho_{tu}(Y) = \rho_{tv}(Y), \qquad \text{for all } Y \in L^p(\mathcal{F}_u), \qquad t \leq u \leq v$$

Relationship Fully-dynamic vs dynamic risk measures

A *fully-dynamic risk measure with restriction property corresponds one-to-one with a dynamic risk measure*:

$$\rho_r(Y) = \rho_{tT}(Y) = \rho_{tu}(Y), \qquad \text{for all } Y \in L^p(\mathcal{F}_u), \qquad t \le u \le T.$$



2. Time consistency

Definition. A fully-dynamic risk measure $(\rho_{tu})_{t,u}$ is

- order time-consistent if for $s \le t \le u$, $X, Y \in L^p(\mathcal{F}_u)$, we have

$$\rho_{tu}(X) = \rho_{tu}(Y) \quad \Longrightarrow \quad \rho_{su}(X) = \rho_{su}(Y).$$

- weak time-consistent, if for $s \le t \le u$, $X \in L^p(\mathcal{F}_u)$,

$$\rho_{su}(X) = \rho_{su}(\rho_{tu}(0) - \rho_{tu}(X))$$

- recursive if for $s \le t \le u$ and $X \in L^p(\mathcal{F}_u)$, we have

$$\rho_{su}(X) = \rho_{st}(-\rho_{tu}(X)).$$

See Acciaio, Penner (2011), Bielecki, Cialenco, Pitera (2017), Bion-Nadal (2009),...
Weak time consistency appeared in Bion-Nadal, DiNunno (2020) in relation to risk indifference prices.

Proposition. The following statements are equivalent:

i) $(\rho_{tu})_{t,u}$ is *recursive time consistency*.

ii) $(\rho_{tu})_{t,u}$ is *order time-consistent* and

$$\rho_{su}(Y) = \rho_{st}(Y - \rho_{tu}(0)), \qquad s \leq t \leq u, \quad Y \in L^p(\mathcal{F}_t). \tag{4}$$

Proof.

$(i) \implies (ii)$. Recursive implies order time consistency, that is immediate. For the second property, we apply translation invariance since $X \in L^p(\mathcal{F}_t)$.

$(ii) \implies (i)$. Conversely, assume that (ii) is satisfied. Let $Z \in L_p(\mathcal{F}_u)$ and define $Y := \rho_{tu}(0) - \rho_{tu}(Z)$. From translation invariance, we have

$$\rho_{tu}(Y) = \rho_{tu}(0) - [\rho_{tu}(0) - \rho_{tu}(Z)] = \rho_{tu}(Z).$$

Thus, from order time consistency, $\rho_{su}(Y) = \rho_{su}(Z)$. Applying (4) to Y, we get that

$$\rho_{su}(Z) = \rho_{st}(-\rho_{tu}(Z)).$$

Observation. The result before hints that the risk evaluation of 0 has some more prominent role than it was expected.

Consequence: corollary

If all the risk measures $(\rho_{tu})_{t,u}$ are normalized, i.e.

$$\rho_{tu}(0) = 0, \quad t \leq u,$$

then $(\rho_{tu})_{t,u}$ is recursive if and only if it is order time consistent and the restriction property holds.

Further Consequence: sub-corollary

In the case of "classical" dynamic risk measures, under the assumption of normalization i.e.

$$\rho_t(0) = 0,$$

the *order time consistency* corresponds one-to-one to the recursive time consistency.

About recursive time consistency

- Recursivity is a “composition rule”.
- Recursivity is not transferred via normalization.

In fact, if $(\rho_{tu})_{t,u}$ is recursive, then its normalized version

$$\bar{\rho}_{tu}(X) := \rho_{tu}(X) - \rho_{tu}(0), \quad X \in L^p(\mathcal{F})_u, \quad t \leq u,$$

may not be.
Indeed, the values $\rho_{su}(0)$, $\rho_{st}(0)$, and $\rho_{tu}(0)$ are potentially different.

Propositin
A normalized fully-dynamic risk measure $\bar{\rho}_{tu}(X)$, is recursive if and only if

$$\rho_{su}(0) = \rho_{st}(0) + E_Q\big[\rho_{tu}(0)|\mathcal{F}_s\big]$$

About order and weak time consistency

- Order time consistency is transferred via normalization.

Proposition. For fully-dynamic risk measures we have equivalence

i) weak time consistency

ii) order time-consistent.

Proof. Assume (ii) Let $X \in L^p(\mathcal{F}_u)$ and $t \leq u$ be arbitrary. By additivity, we have

$$\rho_{tu}(X) = \rho_{tu}(0 - \rho_{tu}(X)) - \rho_{tu}(0) = \rho_{tu}(\rho_{tu}(0) - \rho_{tu}(X)) =: \rho_{tu}(Y). \tag{5}$$

Then order time-consistency imply that $\rho_{su}(X) = \rho_{su}(Y)$.

Conversely, suppose that $\rho_{tu}(X) = \rho_{tu}(Y)$ holds for some $X, Y \in L^p(\mathcal{F}_u)$ and $t \leq u$. By weak time-consistency

$$\rho_{su}(X) = \rho_{su}(\rho_{tu}(0) - \rho_{tu}(X)) = \rho_{su}(\rho_{tu}(0) - \rho_{tu}(Y)) = \rho_{su}(Y).$$

Remark: Under normalization and restriction, the three time consistencies coincide (classical dynamic risk measures)

3. H-longevity and time-consistency

Once we drop restriction, to allow for the evaluation of horizon risk, we introduce h-longevity as a kind of penalization for using a risk measure non-appropriate for the time window.

Definition. Horizon longevity or h-longevity is

$$\gamma(s,t,u,X) := \rho_{su}(X) - \rho_{st}(X) \geq 0$$

for any $s \leq t \leq u, X \in L^p(\mathcal{F}_t)$.

Proposition (acceptance sets). For a fully-dynamic risk measure $(\rho_{st})_{s,t}$:

(a) Restriction is equivalent to $\mathcal{A}_{su} \cap L^p(\mathcal{F}_t) = \mathcal{A}_{st}$ for any $s \leq t \leq u$.

(b) H-longevity is equivalent to $\mathcal{A}_{su} \cap L^p(\mathcal{F}_t) \subseteq \mathcal{A}_{st}$ for any $s \leq t \leq u$.

Remark.
In presence of h-longevity, there is yet another concept of time consistency:

subrecursive time-consistency (or subrecursivity), i.e.

$$\rho_{st}(-\rho_{tu}(X)) \leq \rho_{su}(X) \quad \text{for any } X \in L^p(\mathcal{F}_u),$$

Proposition.
For a fully-dynamic risk measure $(\rho_{st})_{s,t}$ the following are equivalent:

i) Subrecursive time-consistency

ii) $\mathcal{A}_{su} \subseteq \mathcal{A}_{st} + \mathcal{A}_{tu}$,

Proposition.
We have the following relationships:

i) Weak time-consistency, h-longevity, and $\rho_{tu}(0) \leq 0$ imply subrecursivity

ii) Subrecursivity and $\rho_{tu}(0) \geq 0$ imply h-longevity

4. $(\rho_{st})_{s,t}$ generated by one BSDE

Back to L^2-spaces related to $(\Omega, \mathcal{F}, P)$ where we consider a d-dimensional Brownian motion $(B_t)_{t\in[0,T]}$ and its P-augmented natural filtration $(\mathcal{F}_t)_t$

According to Peng (1997), the solution $(Y_t, Z_t)_{t\in[0,T]}$ of the BSDE

$$Y_t = X + \int_t^T g(s, Z_s)\, ds - \int_t^T Z_s\, dB_s$$

can be seen as an operator depending on the driver g and evaluated at the final condition $X \in L^2(\mathcal{F}_T)$, which turns out to represent the nonlinear expectation:

$$\rho_{st}(X) = \mathcal{E}^g(X|\mathcal{F}_s), \qquad X \in L^p(\mathcal{F}_t).$$

Just a recap:
The relationships among BSDEs, nonlinear expectations, and dynamic risk measures is detailed Peng (1997, 2003) and Rosazza Gianin (2006).
Recall that g does not depend on Y, so that $\mathcal{F}_s$-translation invariance is satisfied, and it is convex to guarantee convexity of the risk measures. See also Barrieu and El Karoui (2009), Jiang /2008).
In general, g is assumed Lipschitz, to guarantee the unicity of the solution. Beyond this case (e.g. quadratic) risk measures have been studied in terms of maximal solutions. See Kobilanski (2000) and also Barrieu and El Karoui (2009).,
If we move beyond the Brownian framework to BSDEs with jumps, we refer, e.g., to Royer (2006), Quenez, Sulem (2013), Laeven, Stadje (2014).

Proposition. The following properties are equivalent:

- $g(t,0) = 0$ for any $t \in [0,T]$;
- each ρ_{st} is normalized
- $(\rho_{st})_{s,t}$ satisfies the restriction property

Note that while normalization refers to a single risk measure ρ_{st}, restriction involves the whole family $(\rho_{st})_{s,t}$.

Corollary. $(\rho_{st})_{s,t}$ with $g(t,0) = 0$ are recursive.

Theorem. For a Lipschitz driver with $g(v,0) \geq 0$ for any v, then h-longevity holds. Furthermore,

$$\gamma(s,t,u,X) = E_{\widetilde{Q}_X}\left[\int_t^u g(v,0)dv|\mathcal{F}_s\right],$$

where $\widetilde{Q}_X \sim P$ is a suitable probability measure depending on X.

Examples.

a) Consider the driver $g(t,z) \equiv a \in \mathbb{R} \setminus \{0\}$. For any $t \in [0,u]$ and $X \in L^2(\mathcal{F}_u)$,

$$\rho_{tu}(X) = E_P \left[-X + \int_t^u a\, ds \middle| \mathcal{F}_t \right] = E_P \left[-X | \mathcal{F}_t \right] + a(u-t).$$

Then $(\rho_{tu})_{t,u}$ is not normalized and does not satisfy the restriction property. Instead, it satisfies h-longevity whenever $a > 0$.

b) Consider $g(t,z) = bz + a$ for any $t \in [0,T], z \in \mathbb{R}^d$, with $a, b \in \mathbb{R} \setminus \{0\}$.

By Girsanov Theorem,

$$-dY_t = (bZ_t + a)dt - Z_t dB_t = adt - Z_t dB_t^Q$$

where $E\left[\frac{dQ}{dP} \middle| \mathcal{F}_t \right] = \exp\left\{ -\frac{1}{2} b^2 t + b \cdot B_t \right\}$. Then we have

$$\rho_{tu}(X) = E_Q \left[-X | \mathcal{F}_t \right] + a(u-t).$$

Then $(\rho_{tu})_{t,u}$ is not normalized and does not satisfy the restriction property. Instead, it satisfies longevity whenever $a = g(t,0) > 0$.

Proof of Proposition:

The following properties are equivalent:

(a) $g(t, 0) = 0$ for any $t \in [0, T]$;

(b) each ρ_{st} is normalized for all $s \leq t$

(c) $(\rho_{st})_{s,t}$ satisfies the restriction property

Note that while normalization refers to a single risk measure ρ_{st}, restriction involves the whole family $(\rho_{st})_{s,t}$.

Proof of Proposition.

(a) $\Rightarrow$ (b). Immediate: driver 0, terminal condition 0 and uniqueness of solution.

(b) $\Rightarrow$ (a). Assume that $\rho_{tu}(0) = 0$ for any $t \leq u$. Hence

$$\int_t^u g(s, Z_s)ds = \int_t^u Z_s dB_s, \quad \text{for any } 0 \leq t \leq u \leq T.$$

Since a continuous martingale and a process of finite variation can be equal only if the martingale is constant[14], then for any t the martingale $M_t(u) = \int_t^u Z_s dB_s$, $u \geq t$, with $M_0 = 0$ is identically equal to 0. Hence $Z_s \equiv 0$ and

$$\int_t^u g(s, Z_s)ds \equiv 0, \quad u \geq t$$

By replacing Z_s and taking derivative, we have$g(u, 0) = 0$ for any u.

[14] (see Prop. 1.2 in Chapter 4 of Revuz, Yor (2013).

(a) $\Rightarrow$ (c)[15]. Let $X \in L^2(\mathcal{F}_u)$ and take $v \geq u$. The corresponding solutions of the BSDE with same driver g are

$$(Y_r^{X,u}, Z_r^{X,u}), \quad (Y_r^{X,v}, Z_r^{X,v}) \quad r \leq u.$$

Since $(Y_r^{X,v}, Z_r^{X,v})$ with

$$Y_r^{X,v} = \begin{cases} Y_r^{X,u}; & r \leq u \\ -X; & u < r \leq v \end{cases} \qquad Z^{X,v}(r,s) = \begin{cases} Z_r^{X,u}; & s \leq u \\ 0; & u < s \leq v \end{cases}$$

is a (unique!) solution of $\rho_{tv}(X)$ when $g(t,0) = 0$, then we see the restriction property.

(c) $\Rightarrow$ (b). Let $X \in L^2(\mathcal{F}_t) \subseteq L^2(\mathcal{F}_T)$. By additivity and restriction,

$$-X + \rho_{tT}(0) = \rho_{tT}(X) = \rho_{tu}(X) = -X + \rho_{tu}(0).$$

Hence $\rho_{tT}(0) = \rho_{tu}(0)$ for any $t \leq u \leq T$. Since ρ_{tu} comes from BSDE, then $\rho_{tt}(0) = 0$. Consequently, $\rho_{tu}(0) = 0$ for any $t \leq u \leq T$.

[15] See Peng (1997)

Proof of Theorem (now with detailed statement):

For Lipschitz driver with $g(v,0) \geq 0$ for $v \in [0,T]$, then h-longevity holds. Furthermore, for all $s \leq u$,

$$\gamma(s,t,u,X) = E_{\widetilde{Q}_X}\left[\int_t^u g(v,0)dv|\mathcal{F}_s\right], s \leq t \leq u, X \in L^p(\mathcal{F}_t)$$

where $\widetilde{Q}_X \sim P$ has density

$$\frac{d\widetilde{Q}_X}{dP} = \exp\left\{-\frac{1}{2}\int_s^u |\Delta_z g(v)|^2 dv + \int_s^u \Delta_z g(v) dB_v\right\}.$$

Here above $\Delta_z g(v) = (\Delta_z^i g(v))_{i=1,\dots,d}$ and

$$\Delta_z^i g(v) \triangleq \frac{g(v,Z_v^u) - g(v,\bar{Z}_v^t)}{d(Z_v^{u,i} - \bar{Z}_v^{t,i})} 1_{\{Z_v^{u,i} \neq \bar{Z}_v^{t,i}\}}.$$

The measure $\widetilde{Q}_X$is interpreted as an *h-longevity premium measure*.

Proof of Theorem.

Let $s \leq t \leq u$ and $X \in L^p(\mathcal{F}_t)$. The risk measures are given by:

$$\begin{aligned} \rho_{st}(X) &= -X + \int_s^t g(v, Z_v^t)dv - \int_s^t Z_v^t dB_v \\ \rho_{su}(X) &= -X + \int_s^u g(v, Z_v^u)dv - \int_s^u Z_v^u dB_v, \end{aligned}$$

Set now the $\mathbb{R}^d$-valued process

$$\bar{Z}_v^t = \begin{cases} Z_v^t; & v \leq t \\ 0; & t < v \leq u \end{cases} \quad ; \quad \widetilde{Z}_v = Z_v^u - \bar{Z}_v^t.$$

Then

$$
\begin{aligned}
&\rho_{su}(X) - \rho_{st}(X) \\
&= \int_s^u [g(v, Z_v^u) - g(v, \bar{Z}_v^t)]dv - \int_s^u \widetilde{Z}_v dB_v + \int_t^u g(v,0)dv \\
&= \int_s^u \Delta_z g(v) \cdot \widetilde{Z}_v dv - \int_s^u \widetilde{Z}_v dB_v + \int_t^u g(v,0)dv. \qquad (6)
\end{aligned}
$$

Furthermore, (6) can be rewritten also as

$$
\delta\rho_s = \Gamma^{t,u} + \int_s^u \Delta_z g(v) \cdot \widetilde{Z}_v dv - \int_s^u \widetilde{Z}_v dB_v, \qquad (7)
$$

where $\delta\rho_s \triangleq \rho_{su}(X) - \rho_{st}(X)$ and $\Gamma^{t,u} \triangleq \int_t^u g(v,0)dv$ is the terminal condition (which depends on t but not on s). We can regard then (7) as a BSDE in itself.

Since $\Gamma^{t,u} \geq 0$ for any t by hypothesis, $\Delta_z g(v) \in \mathbb{H}^2_{[s,u]}(\mathbb{R}^d)$, and by the assumption of g Lipschitz in z, we have[16] $\delta\rho_s \geq 0$ for any $s \leq t$.

[16] Prop. 2.2 of El Karoui, Peng,Quenez (1997).

By applying Girsanov Theorem, (6) becomes

$$
\begin{aligned}
\rho_{su}(X) - \rho_{st}(X) &= \int_s^u \Delta_z g(v) \cdot \widetilde{Z}_v dv - \int_s^u \widetilde{Z}_v dB_v + \int_t^u g(v,0)dv \\
&= -\int_s^u \widetilde{Z}_v dB_v^{\widetilde{Q}_X} + \int_t^u g(v,0)dv,
\end{aligned}
$$

where $B_v^{\widetilde{Q}_X} \triangleq B_v - B_s - \int_s^v \Delta_z g(r)\,dr$, $v \in [s,u]$, is a $\widetilde{Q}_X$-Brownian motion. Hence, by taking the conditional expectation with respect to $\widetilde{Q}_X$,

$$
\begin{aligned}
\rho_{su}(X) - \rho_{st}(X) &= E_{\widetilde{Q}_X}\left[-\int_s^u \widetilde{Z}_v dB_v^{\widetilde{Q}_X} + \int_t^u g(v,0)dv \Big| \mathcal{F}_s\right] \\
&= E_{\widetilde{Q}_X}\left[\int_t^u g(v,0)dv \Big| \mathcal{F}_s\right].
\end{aligned}
$$

Since $g(\cdot, 0 \geq 0)$, it follows that $\rho_{su}(X) - \rho_{st}(X) \geq 0$ and that $\gamma(s,t,u,X) = \rho_{su}(X) - \rho_{st}(X) = E_{\widetilde{Q}_X}\left[\int_t^u g(v,0)dv | \mathcal{F}_s\right]$.

5. Outlook: $(\rho_{st})_{s,t}$ generated by a family of BSDEs

To give even more emphasis to the time horizon, we induce risk measures from *a family of BSDEs with drivers $\mathcal{G} = (g_u)_{u \in [0,T]}$,depending on the time horizon u of ρ_{su}.* Then we have

$$\rho_{tu}(X) = \rho^{\mathcal{G}}_{tu}(X) = \mathcal{E}^{g_u}(X|\mathcal{F}_t), \text{ for any } X \in L^2(\mathcal{F}_u).$$

NB: If, for $u \in [0, T]$, $g_u(r, 0) = 0$ for any r, then $\rho^{\mathcal{G}}_{tu}$ is normalized.

However, $g_u(r, 0) = 0$, for any r, u, does not imply the restriction property.

Proposition

Let $g_u(r, 0) = 0$, for any r, u ,and let $g_u(r, \cdot)$ be continuous in r.
The restriction property holds if and only if g_u is constant in u
(i.e. back to a single BSDE!).

Example. Consider the driver $g(t, z) \equiv a_u \in \mathbb{R} \setminus \{0\}$. Then

First we look at time-consistency.

We say that the family $\mathcal{G} = (g_u)_u$ is increasing when, for any $t \leq u$, $g_t(s, y, z) \leq g_u(s, y, z)$ for any $s \in [0, t], y \in \mathbb{R}, z \in R^d$.

Theorem

Let $(\rho_{tu})_{t,u}$ be induced by the BSDEs with a family of drivers $\mathcal{G} = (g_t)_t$

i) $(\rho_{tu})_{t,u}$ is sub time-consistency if and only if $\mathcal{G}$ is increasing,

ii) $(\rho_{tu})_{t,u}$ is strong time-consistency if and only if $\mathcal{G} = \{g\}$.

Proof. The proof relies on the Comparison Theorem for BSDEs, see El Karoui et al. (1997) and also the Converse Comparison Theorem by Briand et al. (2000).

When it comes to h-longevity, we have the following result.

Proposition

i) If $\mathcal{G}$ is increasing and $\rho_{tu}(0) \geq 0$ for any $t \leq u$, then $(\rho_{tu})_{t,u}$ satisfies h-longevity.

ii) If $\mathcal{G}$ is increasing and $g_t \geq 0$ for any $t \in [0, T]$, then $(\rho_{tu})_{t,u}$ satisfies h-longevity.

In fact, this result relies on the *comparison of two BSDEs on different time horizons* $[0, T_1] \subset [0, T_2]$:

$$Y_t^{T_i} = \xi_i + \int_t^{T_i} g^{T_i}(s, Y_s^{T_i}, Z_s^{T_i})ds - \int_t^{T_i} Z_s^{T_i} dB_s.$$

Theorem: comparison of BSDEs on different horizons

If $g^{T_2}(s, y, z) \geq g^{T_1}(s, y, z)$ for any $s \in [0, T_1], y, z$ and $g^{T_2}(s, y, z) \geq 0$ for any $s \in [T_1, T_2], y, z$ and $\xi_2 \geq \xi_1$, then $Y_t^{T_2} \geq Y_t^{T_1}$ for any $s \in [0, T_1]$ and $Y_t^{T_2} \geq \xi_1$ for any $s \in [T_1, T_2]$.

Examples

a) Consider the driver $g_u(t,z) = a_u$ for any $t \in [0,u], z \in \mathbb{R}^d$, with $a_u \in \mathbb{R} \setminus \{0\}$ depending on the maturity u.

We have that, for any $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T)$,

$$\rho_{tu}(X) = E_P\left[-X|\,\mathcal{F}_t\right] + a_u(u-t).$$

This means that, for $a_u \neq 0$, $(\rho_{tu})_{t,u}$ is not normalized and does not satisfy the restriction property. Instead, it satisfies longevity whenever $a_u > 0$ is increasing in u.

b) Consider now the driver $g_u(t,z) = bz + a_u$ for any $t \in [0,u], z \in \mathbb{R}^d$, with $a_u, b \in \mathbb{R} \setminus \{0\}$ and a_u depending on the maturity u.

It follows that

$$\rho_{tu}(X) = E_Q\left[-X|\,\mathcal{F}_t\right] + a_u(u-t),$$

where $E\left[\frac{dQ}{dP}\middle|\,\mathcal{F}_t\right] = \exp\left\{-\frac{1}{2}b^2 t + b \cdot B_t\right\}$. For $a_u \neq 0$, $(\rho_{tu})_{t,u}$ is not normalized and does not satisfy the restriction property. Instead, it satisfies longevity whenever $a_u > 0$ is increasing in u.

c) Entropic type risk measures

In the one-dimensional case, consider the driver $g_u(t,z) = \frac{b_u}{2} z^2 + a_u(r)$ for any $t \in [0,T], z \in \mathbb{R}$, with $b_u > 0$ and a_u positive function:

$$Y_t = -X + \int_t^T \left[b_u \frac{Z_s^2}{2} + a_u \right] ds - \int_t^T Z_s dB_s$$

Then we have

$$\rho_{su}(X) = \frac{1}{b_u} \ln \left(E_P \left[\exp(-b_u X) | \mathcal{F}_s \right] \right) + \int_s^u a_u(t) dt.$$

Hence, $(\rho_{tu})_{t,u}$ is not normalized and does not satisfy the restriction property. Instead, it satisfies h-longevity whenever $(a_u)_u$ and $(b_u)_u$ are increasing in u.

6. Outlook: $(\rho_{st})_{s,t}$ generated by BSVIE

Solutions of BSVIEs can be seen as a selection of random variables from a family of BSDEs, see Yong (2013).

Exploration on the use of BSVIE to generate dynamic risk measures. See Yong (2007) (and Agram (2019) for the jump case). We consider

$$Y_t = X + \int_t^T g(t, s, Z(t, s))\, ds - \int_t^T Z(t, s)\, dB_s,$$

where the driver is

$$g : \Omega \times \Delta \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$$

with

$$\Delta \triangleq \{(t, s) \in [0, T] \times [0, T] : s \geq t\}$$

The relationship with a family of BSDEs (parametrised by t):

$$\eta(r;t,X) = X + \int_r^T \bar{g}(v,\zeta(v;t);t)dv - \int_r^T \zeta(v;t)dB_v, \quad r \in [t;T]$$

where

$$\zeta(v;t) = Z(t,v); \quad \bar{g}(v,\zeta(v;t);t) = g(t,v,Z(t,v)),$$

is given by

$$Y_t = \eta(t;t,X)$$

Remark. Although $\eta(r;t,X)$ and Y_t are closely related, the solution of the BSVIE may satisfy some specific properties since it corresponds not to the whole family $\eta(r;t,X)$ but only to $\eta(t;t,X)$.

Proposition. The following are equivalent:

i) $g(t,u,0) = 0$ for any $0 \leq t \leq u \leq T$;

ii) $(\rho_{tu})_{t,u}$ are normalized

iii) the restriction property holds.

Comment. Hence we will not assume $g(t,u,0) = 0$ to allow for discussion on horizon risk and h-longevity.

About time-consistency.

Theorem. Let $(\rho_{tu})_{t,u}$ be induced by a BSVIE with driver g.

i) we have sub time-consistency if and only if the driver $g(t, v, z)$ is decreasing in t (i.e., for any $s \leq t$, $g(t, v, z) \leq g(s, v, z)$ for any $v \in [t, u]$ and $z \in \mathbb{R}^d$).

ii) we have strong time-consistency if and only if the driver $g(t, \cdot, \cdot)$ is constant in t. (Back to BSDE case!)

Theorem: converse comparison for BSVIE. Let Y_t^1 and Y_t^2 be two BSVIEs with drivers g^1 and g^2 and same terminal condition ξ_t. Let η^1 and η^2 be the corresponding families of BSDEs.
Assume that $g^i(\cdot, v, \cdot)$ is continuous in v for $i = 1, 2$.

i) If $\eta^1(r; t, \xi_t) \leq \eta^2(r; t, \xi_t)$ for any $t \in [0, T]$, $r \in [t, T]$ and $\xi_t \in L^2(\mathcal{F}_T)$, then $g^1 \leq g^2$, that is $g^1(t, v, z) \leq g^2(t, v, z)$ for any $t \in [0, T], v \in [t, T], z \in \mathbb{R}^d$.

ii) If $Y_t^{1,X} \leq Y_t^{2,X}$ for any $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T)$ and if $g^i(t, v, 0) = 0$ for $i = 1, 2$, then $g^1 \leq g^2$.

About h-longevity.

Proposition.
If $g(s, v, 0) \geq 0$ for any $s \leq v$, then longevity holds.

Furthermore,

$$\gamma(s, t, u, X) = E_{\widetilde{Q}_{s,X}}\left[\int_t^u g(s, v, 0)dv \Big| \mathcal{F}_s\right], \qquad s \leq t \leq u,$$

where $\widetilde{Q}_{s,X\sim P}$ is a suitable probability measure depending on X.

Proof. The proof is based on the Comparison Theorem for BSVIEs, see Wang and Yong (2015). Moreover, the representation of the h-longevity is obtained through the study of the dynamics of the difference of BSVIES and an adequate change of measure.

Examples: Entropic type risk measures.

The measure is given by

$$\begin{aligned} Y_t &= -X + \int_t^T \left[b(t)\frac{(Z(t,s))^2}{2} + a(t,s) \right] ds - \int_t^T Z(t,s) dB_s \\ &= -X + \int_t^T a(t,s) ds + \int_t^T b(t) \frac{(Z(t,s))^2}{2} ds - \int_t^T Z(t,s) dB_s^{\widetilde{Q}_t} \end{aligned}$$

Hence,

$$\begin{aligned} Y_t &= \frac{1}{b(t)} \ln E_P \left[\exp\left\{ -b(t)X + \int_t^T a(t,s) ds \Big| \mathcal{F}_t \right\} \right] \\ &= \frac{1}{b(t)} \ln E_P \left[e^{-b(t)X} \Big| \mathcal{F}_t \right] + \int_t^T a(t,s) ds, \end{aligned}$$

that is a translation of the usual entropic risk measure.
Here $b(t) > 0$ and $a(t,s)$ are deterministic. With $a(t,s)0 >$ there is h-longevity.

7. Long time horizons and money

The value of money varies over long time horizons, with the result that financial risk assessment are also subject to the uncertainty on interest rates.

To consider the combination of the two factors, we enter the domain of cash non-additive risk measure.

Quantities expressed in unit of money and $€_t$ is the unit of money at time t. Hence a financial investment available at time t is denoted $X\,€_t$, where X represents the size of the investment.

Let $(D_{st})_{0\leq s\leq t\leq T}$ be the family of discount factors D_{st} on the time interval $(s,t]$:

$$0 < d_{st} \leq D_{st}\,€_t \leq 1.$$

The unit of measurement for D_{st} is $1/€_t$.

For any cash additive fully-dynamic risk measure $(\varphi_{st})_{0\leq s\leq t\leq T}$ we define

$$\rho_{st}(X) \triangleq \varphi_{st}(D_{st}X\,\text{€}_t), \qquad X \in L^p(\mathcal{F}_t).$$

Indeed, ρ_{st} is cash subadditive. For any $X \in L^p(\mathcal{F}_t)$ and $m \in L^p_+(\mathcal{F}_s)$, we have

$$\begin{aligned}\rho_{st}(X+m) &= \varphi_{st}(D_{st}(X+m)\,\text{€}_t)\\ &\geq \varphi_{st}(D_{st}X\,\text{€}_t + m\,\text{€}_t) = \varphi_{st}(D_{st}X\,\text{€}_t) - m\\ &= \rho_{st}(X) - m,\end{aligned}$$

thanks to monotonicity.

Another cash subaddtive risk measure generated by the ambiguity of the interest rates is given by:

$$\mathcal{R}_{st}(X) \triangleq \operatorname*{ess\,sup}_{0<d_{st}\leq D_{st}\,\text{€}_t\leq 1} \varphi_{st}(D_{st}X\,\text{€}_t)$$

In the framework of cash non-additive risk measures we can study h-longevity, normalization, and time-consistency.

PART I+II Where are we?

- Risk measures from static point of view, properties
- Dynamic risk measurement: BSDEs
- Dynamic risk evaluation: the horizon and time consistency
- Fully-dynamic risk measures: restriction and normalization
- Time consistency definitions and h-longevity: interrelationships
- Time consistency and h-longevity in the BSDEs' world

PART III

- Monetary evaluations: when the horizon matters
- Subadditive (cash non-additive) risk measures
- hq-entropic risk measures on losses
- h-generalized shortfall
- hq-entropic risk measure and h-generalized shortfall

1. Long time horizons and money

The value of money varies over long time horizons, with the result that financial risk assessment are also subject to the uncertainty on interest rates.

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Indeed, ρ_{st} is cash subadditive. For any $X \in L^p(\mathcal{F}_t)$ and $m \in L^p_+(\mathcal{F}_s)$, we have

$$\begin{aligned}\rho_{st}(X+m) &= \varphi_{st}(D_{st}(X+m)€_t)\\ &= \varphi_{st}(D_{st}X€_t + D_{st}m€_t) \ge \varphi_{st}(D_{st}X€_t) - m\\ &= \rho_{st}(X) - m,\end{aligned}$$

thanks to monotonicity.

Another cash subaddtive risk measure generated by the ambiguity of the interest rates is given by:

$$\mathcal{R}_{st}(X) \triangleq \operatorname*{ess\,sup}_{0<d_{st}\le D_{st}€_t\le 1} \varphi_{st}(D_{st}X€_t)$$

In the framework of cash non-additive risk measures we can study h-longevity, normalization, and time-consistency.

2. Cash non-additivity and BSDEs

In a dynamic setting, we can generate cash non-additive risk measures from BSDEs with explicit dependence on Y in the driver:

$$Y_t = X + \int_t^u g(s, Y_s, Z_s)\, ds - \int_t^u Z_s\, dB_s$$

- In particular, if $g(s, y, z)$ is decreasing in y for all (s, z), then ρ_{tu} is cash subadditive[17].

Proposition: characterization of restriction and normalization

- ρ_{tu} has **restriction** if and only if $g(t, y, 0) = 0$ for a.a. t, y.
- If g Lipschitz or if $g(t, y, z) = a_t + f(y)|z|^2$ with f continuous, then ρ_{tu} is **normalized** if and only if $g(t, 0, 0) = 0$.

[17] See El Karoui, Ravanelli 2009)

Theorem for g Lipschitz: characterization of H-longevity

H-longevity holds **if and only if** $g(t, y, 0) \geq 0$ for any t, y. Furthermore, we have

$$\gamma(s,t,u,X) = E_{\widetilde{Q}_X}\left[e^{\int_s^u \Delta_y g(v)dv} \int_t^u g(v,-X,0)dv|\mathcal{F}_s\right], \quad s \leq t \leq u, X \in L^p(\mathcal{F}_t),$$

where $\widetilde{Q}_X \sim P$ is a suitable probability measure depending on X and

$$\Delta_y g(v) \triangleq \frac{g(v,Y_v^u,Z_v^u) - g(v,\bar{Y}_v^t,Z_v^u)}{Y_v^u - \bar{Y}_v^t} 1_{\{Y_v^t \neq \bar{Y}_v^u\}} \qquad \bar{Y}_v^t = \begin{cases} Y_v^t; & v \leq t \\ -X; & t < v \leq u \end{cases}$$

The probability $\widetilde{Q}_X$ can be interpreted as an **h-longevity premium measure.**[18]

[18] Di Nunno, Rosazza Gianin (2024).

- This result is now presented as an **if and only if**, full characterization. Indeed, we can use result of BSDEs of the comparison type:
  Comparison theorem (already seen). Under the standard assumption. Consider two solutions assositated to the generators g_i, $i = 1, 2$ for the terminal conditions $\xi_i \in L^2(\mathcal{F}_u)$:

$$Y_t^i = X_i + \int_t^u g_i(s, Y_s^i, Z_s^i)\, ds - \int_t^u Z_s^i\, dW_s, \qquad 0 \le t \le T, \quad i = 1, 2.$$

  Assume that

$$X_1 \le X_2 \quad \mathbb{P}\text{-a.s.}$$

  and that, for all (t, ω) and all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$g_1(\omega, t, y, z) \le g_2(\omega, t, y, z).$$

  Then the corresponding solutions satisfy

$$Y_t^1 \le Y_t^2 \quad \mathbb{P}\text{-a.s. for all } 0 \le t \le T.$$

- **Converse comparison theorem for Lipschitz BSDEs**[19] For each terminal condition $\xi \in L^2(\mathcal{F}_T)$ and $i = 1, 2$, consider the BSDEs with unique solutions:

$$Y_t^i = \xi + \int_t^T g_i(s, Y_s^i, Z_s^i)\, ds - \int_t^T Z_s^i\, dW_s, \qquad 0 \le t \le T,$$

which admits a unique solution $(Y^i, Z^i) \in \mathcal{S}^2 \times \mathcal{H}^2$.

Assume

$$Y_t^1 \le Y_t^2 \quad \mathbb{P}\text{-a.s. for all } t \in [0, T] \text{ and all bounded } \xi \in L^\infty(\mathcal{F}_T).$$

Then the generators are ordered almost everywhere:

$$g_1(t, \omega, y, z) \;\le\; g_2(t, \omega, y, z) \quad \text{for } dt \otimes d\mathbb{P}\text{-a.e. } (t, \omega) \text{ and all } (y, z) \in \mathbb{R} \times \mathbb{R}^d.$$

[19] Briand, Coquet, Hu, Mémin, Peng (2000)

Example.

Consider the driver

$$g(t, y, z) = r_t y^- + z, \quad t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d,$$

where r_t can be interpreted as a positive interest rate depending on time t.

It is immediate to see that $g(\cdot, y, z)$ is decreasing in y, convex and Lipschitz in (y, z). Then we obtain cash subadditivity, normalization (since $g(t, 0, 0) = 0$) and h-longevity (by $g(t, y, 0) \geq 0$ for any t, y.

Theorem

Assume g is quadratic of type (3) with f continuous in y. H-longevity implies that $g(t, y, 0) \geq 0$ for a.a. $t \in [0, T]$ and all $y \in \mathbb{R}$.

3. hq-entropic risk measures on losses

Driven by considerations on capital requirements on potential losses in long term horizons, we consider

$$Y_t = \mathcal{X} + \int_t^u \left[\frac{q}{2} \frac{Z_s^2}{1 + (1-q) Y_s} + a(s) \right] ds - \int_t^T Z_s dB_s$$

The solution of such BSDE is Tsallis generalised entropy

$$Y_t = \ln_q E\left[\exp_q \left(\mathcal{X} + \int_t^u a(s) ds \right) \Big| \mathcal{F}_t \right]$$

given in terms of the **generalised q-exponential** and **q-logarithmic functions** for $q \in (0,1)$:

$$\exp_q(x) = [1 + (1-q)x]^{\frac{1}{1-q}}, \qquad x \geq -\frac{1}{1-q}$$

and

$$\ln_q(x) = \frac{x^{1-q} - 1}{1-q}, \qquad x \geq 0$$

Definition. The **hq-entropic measure on losses** is then

$$\rho_{tu}^{hq}(X) = \ln_q E_P\left[\exp_q\left((X+\beta)^- + \alpha_q + \int_t^u a_s ds\right)\Big|\,\mathcal{F}_t\right],$$

here β represents a level of acceptable loss, while $\alpha_q \geq \frac{1}{q-1}$ and $q \in (0,1)$ provide how conservative the risk evaluation should be.

This risk measure is **convex**, **cash subadditive**, **NOT normalized**, **NOT restricted**, and there is **h-longevity** whenever $a_s > 0$.

Remark. Take $a \equiv 0$. For a fixed q, the higher is the value α_q, the more conservative is the corresponding measure $\rho_{tu}^q(X)$. In fact,

$$\alpha_q^1 \leq \alpha_q^2 \quad \Longrightarrow \quad \ln_q E\left[\exp_q((X+\beta)^- + \alpha_q^1)\big|\,\mathcal{F}_t\right] \;\leq\; \ln_q E\left[\exp_q((X+\beta)^- + \alpha_q^2)\big|\,\mathcal{F}_t\right].$$

Proposition. Take $a \equiv 0$. For any $X \in L^2(\mathcal{F}_u), \beta \in \mathbb{R}$, the q-entropic risk measure on losses ρ_{tu}^q is increasing in q and $\alpha_0 \leq \alpha_1 \leq \alpha_1$ and

$$E_P[(X+\beta)^- + \alpha_0|\mathcal{F}_t] = \rho_{tu}^0(X) \;\leq\; \rho_{tu}^q(X) \;\leq\; \rho_{tu}^1(X) = \ln E_P[\exp(X+\beta)^- + \alpha_1|\mathcal{F}_t].$$

4. H-Generalized shortfall

The **classical entropic risk measure**[20] can be regarded as a shortfall risk measure, that is

$$\rho_{tu}(X) = \operatorname{ess\,inf}\left\{m_t\ L^p(\mathcal{F}_t):\ E\left[U(X+m_t)|\mathcal{F}_t\right] \geq B\right\}$$

with U utility function and B a fixed target level.

We study the extension of shortfall risk measures to capture horizon risk.

Definition A fully-dynamic shortfall risk measure is the family $(\rho_{tu}^{U,B})_{t,u}$:

$$\rho_{tu}^{U,B}(X) = \operatorname{ess\,inf}\{m_t \in L^p(\mathcal{F}_t) : E[\,U_u(X+m_t)|\,\mathcal{F}_t] \geq B_{tu}\}, \quad X \in L^p(\mathcal{F}_u),$$

generated by the families of real numbers $(B_{tu})_{t,u}$ and of mappings $(U_u)_u$ with concave and non-decreasing $U_u : [-\infty, +\infty) \longrightarrow [-\infty, +\infty)$.

[20] Föllmer and Schied (2011)

H-entropic risk measure

. Consider $U_u(x) = 1 - e^{-x+\int_0^u a_s ds}$, $x \in \mathbb{R}$, with $a_s \geq 0$ and

$$B_{tu} = 1 - e^{\int_0^t a(s)ds}.$$

Then we have the h-entropic risk measure ($b = 1$):

$$\rho_{tu}(X) = \ln\left(\frac{E[e^{-X+\int_0^u a(s)ds}|\mathcal{F}_t]}{1 - B_{tu}}\right) = \ln E[e^{-X+\int_t^u a(s)ds}|\mathcal{F}_t].$$

Proposition If $(U_u - B_{tu})_{t,u}$ is non-increasing in u for all t, then the h-longevity of $(\rho_{tu})_{t,u}$ holds. Whenever both U_u and B_{tu} are constant in u, restriction holds.

H-Value-at-Risk.

The shortfall representation of Value at Risk is, for $\alpha \in (0,1)$:

$$VaR_\alpha(X) = \inf\{m \in \mathbb{R} : E[U^\alpha(X+m)] \geq 0\},$$

with $B = 0$ and $U^\alpha(x) = (\alpha - 1)1_{\{x<0\}} + \alpha 1_{\{x \geq 0\}}$.

Then we consider $U_u^\alpha(x) = (\alpha_u - 1)1_{\{x<0\}} + \alpha_u 1_{\{x\geq 0\}}$, with $\alpha_u \in (0,1)$ depending on the time horizon u. Then,

$$\begin{aligned}\rho_{tu}(X) &= \operatorname{ess\,inf}\{m_t \in L^p(\mathcal{F}_t) : E[\,U_u^\alpha(X+m_t)|\,\mathcal{F}_t] \geq B_{tu}\} \\ &= \operatorname{ess\,inf}\{m_t \in L^p(\mathcal{F}_t) : \alpha_u - 1 + P(\,X + m_t \geq 0|\,\mathcal{F}_t) \geq B_{tu}\}.\end{aligned}$$

Taking $B_{tu} \equiv 0$, it follows that

$$\rho_{tu}(X) = \operatorname{ess\,inf}\{m_t \in L^p(\mathcal{F}_t) : P(\,X + m_t \geq 0|\,\mathcal{F}_t) \geq 1 - \alpha_u\} \triangleq VaR_{t,u,\alpha_u}(X).$$

By Proposition before, h-longevity is guaranteed when α_u is decreasing in u.

H-shortfall and interest rates uncertainty

To capture cash nonadditivity, we need to extend further the concept of shortfall.

Definition Given a family $(f_u)_{u\in[0,T]}$ of functions $f_u : \mathbb{R}^2 \to [-\infty, +\infty)$, a family $(U_u)_{u\in[0,T]}$ of concave non-decreasing utility functions $U_u : [-\infty, +\infty) \longrightarrow [-\infty, +\infty)$, and a family of deterministic targets $(B_{tu})_{0\le t\le u\le T}$, we define the h-generalized shortfall risk measure $(\rho_{tu}^{U,f,B})_{t,u}$ as

$$\rho_{tu}^{U,f,B}(X) = \operatorname{ess\,inf}\{m_t \in L^p(\mathcal{F}_t) : E[U_u(f_u(X, m_t))|\mathcal{F}_t] \ge B_{tu}\}.$$

The corresponding family of acceptance sets is then given by

$$\mathcal{A}_{tu}^{U,f,B;m_t} = \{Y \in L^p(\mathcal{F}_u) : E[U_u(f_u(Y, m_t))|\mathcal{F}_t] \ge B_{tu}\}.$$

which depend on $(m_t)_{m_t\in L^p(\mathcal{F}_t)}$, as the measures are cash nonadditive:

$$\rho_{tu}^{U,f,B}(X) = \operatorname{ess\,inf}\{m_t \in L^p(\mathcal{F}_t) : X \in \mathcal{A}_{tu}^{U,f,B;m_t}\}, \quad X \in L^p(\mathcal{F}_u).$$

Proposition

a) If $(U_u \circ f_u)(y, m)$ is increasing in y and m, and concave in y, then $\rho_{tu}^{U,f,B}$ is quasi-convex:

$$\rho_{tu}^{U,f,B}(\lambda X + (1-\lambda)Y) \leq \max\left\{\rho_{tu}^{U,f,B}(X); \rho_{tu}^{U,f,B}(Y)\right\} \qquad (\lambda \in [0,1]).$$

b) If $(U_u \circ f_u)(y, m)$ is also concave in (y, m), then $\rho_{tu}^{U,f,B}$ is also convex.
c) Cash subadditivity is verified under some growth assumptions on f_u.

5. Hq-entropic on losses and relation with utilities

Recall

$$\rho_{tu}^{hq}(X) = \ln_q E\Big[\exp_q\left((X+\beta)^- + \alpha_q + A(t,u)\right)\Big|\,\mathcal{F}_t\Big],$$

with $A(t,u) \geq 0$, for all $t \leq u$, increasing in u.

Theorem The hq-entropic risk measure on losses is a h-generalized shortfall with representation via $(U_u, f_u, B_{tu})_{t,u}$ satisfying:

$$U_u(f_u(y,m)) - B_{tu} = \exp_q(m) - \exp_q\left((y+\beta)^- + \alpha_q + A(t,u)\right).$$

In particular, we can choose $U_u(x) = x$ and

$$\begin{aligned} f_u(y,m) &= B_{tu} + \exp_q(m) - \exp_q((y+\beta)^- + \alpha_q + A(t,u)) \\ &= B_{tu} + \exp_q(m)\left[1 - \exp_q\left(\frac{(y+\beta)^- + \alpha_q + A(t,u) - m}{1+(1-q)m}\right)\right] \end{aligned}$$

for some target $(B_{tu})_{t,u}$.

PART IV

- Computational aspects - Lipschitz case
- Solution operator: Wiener chaos approach
- An operator Euler scheme
- Finite-dimensional approximation
- Deep Operator BSDE - numerical example

1. Computational aspects: evaluation vs operator

NOTE: Here we consider dynamic case with T horizon.

Recall that, at t, the process $(Y_t)_t$ of the solution $(Y_t, Z_t)_t$ of

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s \cdot dB_s, \qquad (*)$$

can be seen as an operator:

$$\mathcal{E}^g(\xi \mid \mathcal{F}_t) = Y_t, \qquad \xi \in L^2(\mathcal{F}_T).$$

for fixed g and framework $\big(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_t, (B_t)_t\big)$, where the information flow is the natural augmented filtration of the Brownian motion.

- Classical numerical methods for BSDEs are established for the calculation of $\mathcal{E}^g(\xi|\mathcal{F}_t)$ for a fixed ξ.
- Instead, we will propose a numerical method to approximate the entire operator.

Indeed, approximations for a "fixed terminal condition" imply that whenever one needs to evaluate the solution at a different position, the algorithm needs to be re-executed from scratch.

The operator solution instead provides an immediate evaluation. This is particularly interesting when we wish to evaluate the risks with (path-wise) functional different types.

Standard assumptions on BSDEs

1. There exists $L > 0$ such that for all $t \in [0, T]$

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|) \quad \forall (y_1, z_1), (y_2, z_2).$$

2. $\xi \in L^2(\mathcal{F}_T)$.

Theorem: Well-posedness.

The BSDE with standard data (ξ, g) has a unique solution $(Y, Z) \in \mathbb{H}_2(\mathbb{R}) \times \mathbb{H}_2(\mathbb{R}^d)$, where

$$\|\varphi\|^2_{\mathbb{H}_2} := \mathbb{E}\Big[\int_0^T |\varphi_t|^2 dt\Big] < \infty$$

Theorem: Lipschianity

Let g satisfy the standard assumption.

Then the solution operators

$$\begin{array}{rccc} \mathcal{Y}: & L^2(\mathcal{F}_T) & \to & \mathbb{H}_2(\mathbb{R}) \\ & \xi & \mapsto & Y \end{array}$$

$$\begin{array}{rccc} \mathcal{Z}: & L^2(\mathcal{F}_T) & \to & \mathbb{H}_2(\mathbb{R}^d) \\ & \xi & \mapsto & Z \end{array}$$

are well-defined and they are globally Lipschitz.

2. Solution operators: a Wiener chaos expansion approach

Theorem: Wiener chaos expansion. Any random variable $\xi \in L^2(\mathcal{F}_T)$, admits representation by means of an orthonormal system in the form:

$$\xi = \sum_{a \in \mathcal{A}} d_a \times \prod_{i \geq 1} H_{a_i}\Big(\int_0^T h_i(s) \cdot dB_s \Big),$$

where

- $\mathcal{A} = \{(a_1, a_2, \dots), a_i \in \mathbb{N} \cup \{0\}, \sum a_i < \infty\}$, i.e. in all the sequences only a finite number of terms does not vanish;
- $(h_i)_{i \geq 1}$ orthonormal basis of $L^2([0, T]; \mathbb{R}^d)$;
- $d_a = a! \, \mathbb{E}\Big[\xi \times \prod_{i \geq 1} H_{a_i}\Big(\int_0^T h_i(s) \cdot dB_s \Big)\Big]$, where $a! := \prod a_i!$
- H_n Hermite polynomial of order n:

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\big(e^{\frac{x^2}{2}}\big).$$

Associated ∞-dim SDEs

$$\xi = \sum_{a\in\mathcal{A}} d_a X_T^{(a_1,a_2,\dots)} \qquad \text{with} \qquad X_t^{(a_1,a_2,\dots)} := \prod_{i\geq 1} H_{a_i}\Big(\int_0^t h_i(s)\cdot dB_s\Big) \in \mathbb{R}^{\mathcal{A}},$$

Proposition [Diaz Lozano, dN '25] $X = (X_t)_t$ satisfies the ∞-dimensional linear SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,$$

where $b\colon [0,T]\times\mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ and $\sigma\colon [0,T]\times\mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}\times d}$ are given by

$$b(t,x)^{(a_1,a_2,\dots)} = \frac{1}{2}\sum_{i\geq 1} x^{(a_1,\dots,a_i-2,\dots)}|h_i(t)|^2_{\mathbb{R}^d} + \sum_{1\leq j<i} x^{(a_1,\dots,a_j-1,\dots,a_i-1,\dots)}\langle h_j(t), h_i(t)\rangle_{\mathbb{R}^d},$$

$$\sigma(t,x)^{(a_1,a_2,\dots)} = \sum_{i\geq 1} x^{(a_1,\dots,a_i-1,\dots)} h_i(t),$$

with $x^a \equiv 0$ for $a_i < 0$ for some $i \in \mathbb{N}$. The initial condition x_0 is determined by $H_n(0) = 0$ for odd n's and $H_{2k}(0) = \frac{(-1)^k}{2^k k!}$ for all $k \geq 1$.

Associated ∞-dim FBSDEs

Then we connect the solution of the BSDE (*) to the solution of the FBSDE system

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

$$Y_t = \sum_{a\in\mathcal{A}} d_a X_T^a + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dB_s.$$

Theorem: Markovian structure.

For every $\xi \in L^2(\mathcal{F}_T)$, there exist $u, v \in L^2([0,T] \times \mathbb{R}^{\mathcal{A}}; \nu)$ such that

$$Y_t = u(t, X_t), \quad Z_t = v(t, X_t),$$

where ν is the law of $(X_t)_t$ on $[0,T] \times \mathbb{R}^{\mathcal{A}}$.

This provides two ways of looking at the solution operators of the BSDE:

$$\mathcal{Y}: \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & \mathbb{H}_2(\mathbb{R}) \\ \xi & \mapsto & Y \end{array}, \qquad \mathcal{Z}: \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & \mathbb{H}_2(\mathbb{R}^d) \\ \xi & \mapsto & Z \end{array}$$

or

$$\mathcal{Y}: \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & L^2([0,T] \times \mathbb{R}^{\mathcal{A}}, \nu) \\ \xi & \mapsto & u \end{array}, \qquad \mathcal{Z}: \begin{array}{ccc} L^2(\mathcal{F}_T) & \to & L^2([0,T] \times \mathbb{R}^{\mathcal{A}}, \nu) \\ \xi & \mapsto & v \end{array}.$$

Also, the operators

$$\begin{array}{rccc} \mathcal{Y}: & L^2(\mathcal{F}_T) & \to & L^2([0,T]\times\mathbb{R}^{\mathcal{A}},\nu) \\ & \xi & \mapsto & u \end{array}, \qquad \begin{array}{rccc} \mathcal{Z}: & L^2(\mathcal{F}_T) & \to & L^2([0,T]\times\mathbb{R}^{\mathcal{A}},\nu) \\ & \xi & \mapsto & v \end{array}$$

are equivalent to the functionals

$$\begin{array}{rccc} \mathscr{Y},\mathscr{Z}: & [0,T]\times\mathbb{R}^{\mathcal{A}}\times L^2(\mathcal{F}_T) & \to & \mathbb{R}\times\mathbb{R}^d \\ & (t,x,\xi) & \mapsto & \big(\mathcal{Y}(\xi)(t,x),\mathcal{Z}(\xi)(t,x)\big) \end{array}$$

and we have the following result

Proposition: Joint measurability

$\mathscr{Y},\mathscr{Z}: [0,T]\times\mathbb{R}^{\mathcal{A}}\times L^2(\mathcal{F}_T)\to\mathbb{R}\times\mathbb{R}^d$ are jointly measurable with respect to the product σ-algebra.

3. An Operator Euler scheme for BSDEs

Set $\pi := \{t_i\}_{0 \le i \le n} \subset [0, T]$. Then

$$Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} g(t, Y_t, Z_t)dt - \int_{t_i}^{t_{i+1}} Z_t \cdot dB_t \approx Y_{t_{i+1}} + \Delta t_i \times g(t_i, Y_{t_i}, Z_{t_i}) - Z_{t_i} \cdot \Delta B_i.$$

Classical (implicit) Euler scheme:

$$Z_i^\pi := \frac{1}{\Delta t_i} \mathbb{E}_{t_i}\left[Y_{i+1}^\pi \Delta B_i\right], \quad Y_i^\pi = \mathbb{E}_{t_i}\left[Y_{i+1}^\pi\right] + \Delta t_i \times g(t_i, Y_i^\pi, Z_i^\pi), \quad Y_n^\pi := \xi.$$

$Z_i^\pi, Y_i^\pi \in L^2(\mathcal{F}_{t_i})$ are random variables, being $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$.

3. An Operator Euler scheme for BSDEs

Set $\pi := \{t_i\}_{0 \le i \le n} \subset [0, T]$. Then

$$Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} g(t, Y_t, Z_t)dt - \int_{t_i}^{t_{i+1}} Z_t \cdot dB_t \approx Y_{t_{i+1}} + \Delta t_i \times g(t_i, Y_{t_i}, Z_{t_i}) - Z_{t_i} \cdot \Delta B_i.$$

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$Z_i^\pi, Y_i^\pi \in L^2(\mathcal{F}_{t_i})$ are random variables, being $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$.

Operator Euler scheme:

$$\mathcal{Z}_i^\pi(\xi) := \frac{1}{\Delta t_i} \mathbb{E}_{t_i}\left[\mathcal{Y}_{i+1}^\pi(\xi) \Delta B_{t_i} \right], \quad \mathcal{Y}_i^\pi(\xi) = \mathbb{E}_{t_i}\left[\mathcal{Y}_{i+1}^\pi(\xi) \right] + \Delta t_i \times g(t_i, \mathcal{Y}_i^\pi(\xi), \mathcal{Z}_i^\pi(\xi)), \quad \mathcal{Y}_n^\pi(\xi) := \xi.$$

$\mathcal{Z}_i^\pi, \mathcal{Y}_i^\pi : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_{t_i})$ are operators recursively given with $\mathcal{Y}_n^\pi(\xi) := \xi$, $\mathcal{Z}_n^\pi(\xi) := 0$.

Convergence of the Operator Euler scheme for BSDEs

To measure the regularity of the solution operator of the BSDE, we define, for $\pi := \{t_i\}_{0\le i\le n} \subset [0, T]$, the functionals $(\mathcal{R}^Y, \mathcal{R}^Z)(\cdot, \pi) \colon L^2(\mathcal{F}_T) \to [0, \infty)$ by

$$\mathcal{R}^Y(\xi, \pi) := \max_{i=0,\dots,n-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} |\mathcal{Y}_t(\xi) - \mathcal{Y}_{t_i}(\xi)|^2 \Big]$$

and

$$\mathcal{R}^Z(\xi, \pi) := \mathbb{E}\Big[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\mathcal{Z}_t(\xi) - \overline{\mathcal{Z}}_i^\pi(\xi)|^2 dt \Big], \qquad \overline{\mathcal{Z}}_i^\pi(\xi) := \frac{1}{\Delta t_i} \mathbb{E}_{t_i}\Big[\int_{t_i}^{t_{i+1}} \mathcal{Z}_t(\xi) dt \Big].$$

Theorem: Convergence of the Operator Euler scheme for BSDEs.

Define the error as

$$\mathcal{E}(\xi,\pi) := \max_{0\le i\le n-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} |\mathcal{Y}_t(\xi) - \mathcal{Y}_i^{\pi}(\xi)|^2\Big] + \mathbb{E}\Big[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |\mathcal{Z}_t(\xi) - \mathcal{Z}_i^{\pi}(\xi)|^2 dt\Big].$$

Then we have:

$$\mathcal{E}(\xi,\pi) \le C\Big(|\pi| + \mathcal{R}^Y(\xi,\pi) + \mathcal{R}^Z(\xi,\pi)\Big).$$

- **Pointwise convergence:** for each ξ, we have $\mathcal{E}(\xi,\pi) \to 0$ when $|\pi| \to 0$
- **Uniform convergence in compact subsets:** if $\pi_n \subset \pi_{n+1}$, then for all compact subsets $K \subset L^2(\mathcal{F}_T)$,

$$\sup_{\xi\in K} \mathcal{E}(\xi,\pi_n) \to 0 \quad \text{when } n \to \infty.$$

Recalling the equivalence between the solution operators $(\mathcal{Y}, \mathcal{Z})$ and some measurable functionals $(\mathscr{Y}, \mathscr{Z})$, we can now express the approximations $(\mathcal{Y}_i^\pi, \mathcal{Z}_i^\pi)$ in terms of X_{t_i}.

We prove an equivalent version for the approximation $(\mathcal{Y}_i^\pi, \mathcal{Z}_i^\pi)$.

Theorem

For $i = 0, \ldots, n-1$, there exist measurable functionals $\mathscr{Y}_i^\pi, \mathscr{Z}_i^\pi : \mathbb{R}^{\mathcal{A}} \times L^2(\mathcal{F}_T) \to \mathbb{R} \times \mathbb{R}^d$ such that

$$\mathcal{Y}_i^\pi(\xi) = \mathscr{Y}_i^\pi(X_{t_i}, \xi), \quad \mathcal{Z}_i^\pi(\xi) = \mathscr{Z}_i^\pi(X_{t_i}, \xi) \quad \mathbb{P} - a.s.$$

In the implementation, we will have to compute these functionals $\mathscr{Y}_i^\pi, \mathscr{Z}_i^\pi$.

4. Finite-dimensional approximation

The first step is the dimension.

Goal: to approximate $\mathscr{Y}_i^\pi, \mathscr{Z}_i^\pi : \mathbb{R}^{\mathcal{A}} \times L^2(\mathcal{F}_T) \to \mathbb{R} \times \mathbb{R}^d$ with maps over a finite-dimensional domain.

Theorem: Convergence of the finite-dimensional approximation

Let $p, M \in \mathbb{N}$. Then, for all $\xi \in L^2(\mathcal{F}_T)$, we have that

$$\max_{0 \le i \le n} \mathbb{E}\Big|\mathscr{Y}_i^\pi(X_{t_i}, \xi) - (\mathscr{Y}_i^\pi \circ \Pi)(X_{t_i}, \xi)\Big|^2 + \sum_{i=0}^{n-1} \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |\mathscr{Z}_i^\pi(X_{t_i}, \xi) - (\mathscr{Z}_i^\pi \circ \Pi)(X_{t_i}, \xi)|^2 dt\Big] \le C\mathbb{E}|\xi - \Pi_{p,M}(\xi)|^2.$$

Comments

- Let $p \in \mathbb{N}$ denote the maximum order of the chaos decomposition, and $M \in \mathbb{N}$ the number of elements in the truncated basis of $L^2([0,T];\mathbb{R}^d)$. For $\xi \in L^2(\mathcal{F}_T)$, we define its projection

$$\Pi_{p,M}(\xi) := \sum_{k=0}^{p} \sum_{|a|=k} d_a X_T^a, \quad a = (a_1, \ldots, a_M). \tag{8}$$

  Recall that each $a = (a_1, \ldots, a_M)$ with $|a| = k$ means that $\sum_{j=1}^{M} a_j = k$.

  The dimension of $\Pi_{p,M}(L^2(\mathcal{F}_T))$ is finite.

- When we restrict the terminal condition ξ to $\Pi_{p,M}(L^2(\mathcal{F}_T))$, the ∞–dim FBSDE is equivalent to a finite-dimensional one, since the terminal condition only depends on a finite number of dimensions of the forward SDE.

5. Deep Operator BSDE - numerical example

Due to the high dimensionality of the projection, we choose **to use neural networks to approximate the maps** $(\mathscr{Y}_i^\pi \circ \Pi, \mathscr{Z}_i^\pi \circ \Pi)_{i=0,\dots,n-1}$.

We call such method the **Deep Operator BSDE.**

- We first fix the generator g, and train the method on 28 families of terminal conditions, each one with $1-3$ parameters.
- We approximate the coefficients of the truncated chaos expansion for several terminal conditions belonging to this subset, from which we keep the maximum and the minimum along each $a \in \mathcal{A}_{p,M}$ to find the coefficients and the compact subset for approximating the solution operator.
- Note that the method does not see the individual families during training, which underlines that the learned solution operator is genuinely defined on a much higher dimensional domain.

- We choose to model $(\mathscr{Y}_i^\pi, \mathscr{Z}_i^\pi)$ with the Multilevel Neural Networks presented in Julius Berner et. al. (2020)[21] . The number of parameters is around 2.8M.
- We choose $p = 3$ and $M = 5$. The time partition π for the Euler scheme is $\pi = \{i \times \frac{T}{10} : 0 \leq i \leq 10\}$.

  We set the dimension of the Brownian motion to $d = 2$. The number of chaos coefficients is 286.
- As for the baseline, we use the method proposed in Briand and Labart (2014)[22].

  NOTE that these baseline methods evaluate the solution operators at a particular terminal condition, while our proposed methodology learns the whole solution operator.

[21] J. Berner et. al. “Numerically solving parametric families of high-dimensional kolmogorov partial differential equations via deep learning”, 2020

[22] P. Briand and C. Labart. “Simulation of BSDEs by Wiener Chaos Expansion”. Annals Appl. Probab. 24.3 (2014), pp. 1129–1171.

Numerical results

We consider the case of pricing and hedging of options in a Black-Scholes setting of dimension 2. The dynamics are given by

$$S_t^j = s_0^j e^{(\mu^j - (\sigma^j)^2/2)t + \sigma^j B_t^j}, \quad \forall t \in [0, T], \quad j = \{1, 2\}, \quad \langle B^1, B^2 \rangle_t = \rho t.$$

If one assumes that the borrowing rate R is higher than the lending rate r, pricing and hedging an option ξ is equivalent to solving a BSDE with terminal condition ξ and nonlinear generator g defined by

$$g(t, y, z) = -ry - \theta \cdot z + (R - r)\Big(y - (\Sigma^{-1} z)_1 - (\Sigma^{-1} z)_2\Big)_{-}.$$

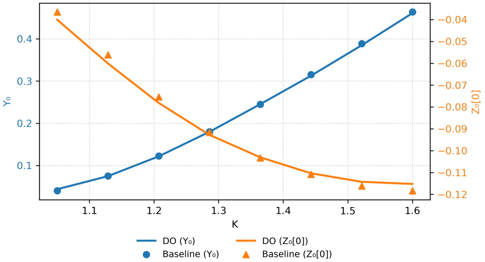
We assess the discrepancy between the Deep Operator BSDE and the baseline using the scaled relative error

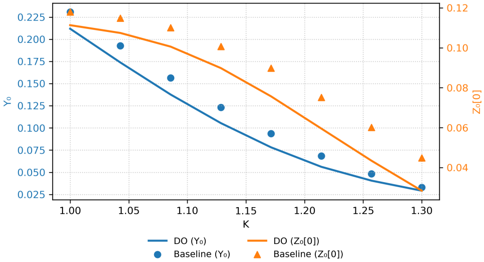
$$\mathrm{err}_Y(\xi) = \frac{|\mathcal{Y}_0^{\mathrm{DO}}(\xi) - \mathcal{Y}_0^{\mathrm{Base}}(\xi)|}{1 + |\mathcal{Y}_0^{\mathrm{Base}}(\xi)|},$$

with an analogous definition for Z.

Option	$\mathbf{err}_Y$ ($\times 10^{-3}$)	$\mathbf{err}_Z$ ($\times 10^{-2}$)
Call – Single	2.21 ± 0.65	0.41 ± 0.06
Call – Basket Weighted	2.58 ± 0.74	0.41 ± 0.07
Call – Spread	7.54 ± 1.08	0.25 ± 0.04
Call – Max	3.66 ± 0.42	0.58 ± 0.07
Call – Min	1.75 ± 0.98	0.32 ± 0.03
Call – Geometric	2.56 ± 0.73	0.41 ± 0.06
Call – Ratio	6.28 ± 1.02	0.25 ± 0.04
Put – Single	3.60 ± 0.16	0.24 ± 0.09
Put – Basket Weighted	3.19 ± 0.28	0.26 ± 0.09
Put – Spread	7.53 ± 1.09	0.27 ± 0.04
Put – Max	1.43 ± 0.38	0.18 ± 0.04
Put – Min	1.38 ± 0.49	0.38 ± 0.10
Put – Geometric	3.08 ± 0.27	0.26 ± 0.09
Put – Ratio	6.52 ± 1.02	0.35 ± 0.06
Asian Call – Single	11.00 ± 0.76	1.33 ± 0.07
Asian Put – Single	7.07 ± 1.39	0.20 ± 0.03
Asian Call – Basket Weighted	10.87 ± 0.80	1.21 ± 0.10
Asian Put – Basket Weighted	8.01 ± 1.35	0.18 ± 0.04
Asian Call – Spread	2.13 ± 0.66	1.01 ± 0.08
Asian Put – Spread	2.17 ± 0.64	1.00 ± 0.08
Asian Call – Max	13.06 ± 0.83	1.58 ± 0.08
Asian Put – Max	9.53 ± 1.16	0.30 ± 0.06
Asian Call – Min	3.22 ± 1.11	0.98 ± 0.09
Asian Put – Min	3.64 ± 1.14	0.28 ± 0.13

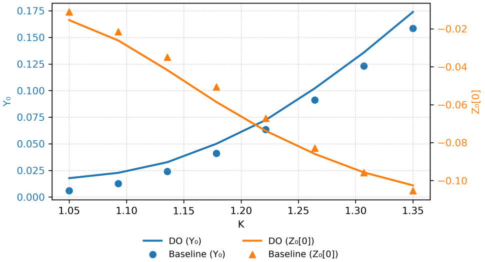
Table: Summary of some option families and their errors (mean $\pm$ std).

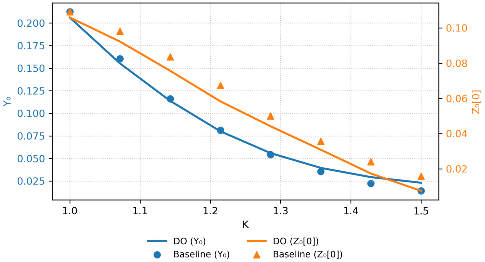




Left: Put – Max options, $\xi(K) = (K - \max(S_T^1, S_T^2))_+, \ \ K \in [1.05, 1.60]$.

Right: Asian Call – Max options, $\xi(K) = \left(\frac{1}{10}\sum_{j=1}^{10} \max(S_{t_j}^1, S_{t_j}^2) - K\right)_+, \ \ K \in [1.00, 1.30]$.





Left: Asian Put – Max options, $\xi(K) = \left(K - \frac{1}{10}\sum_{j=1}^{10} \max(S^1_{t_j}, S^2_{t_j})\right)_+, \quad K \in [1.05, 1.35]$.

Right: Call – Max options, $\xi(K) = (\max(S^1_T, S^2_T) - K)_+, \quad K \in [1.00, 1.50]$.

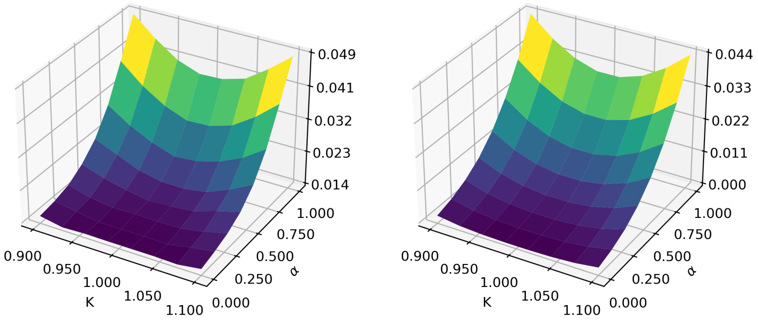


Figure: Comparison of Y_0.

Asian Put – Basket weighted options, $\xi(K,\alpha) = \left(K - \frac{1}{10}\sum_{j=1}^{10}(\alpha S^1_{t_j} + (1-\alpha)S^2_{t_j})\right)_+$, $K \in [0.90, 1.10]$, $\alpha \in [0,1]$.

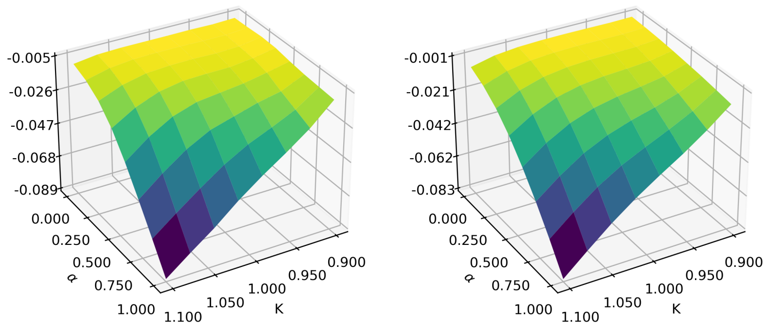


Figure: Comparison of Z_0.

Asian Put – Basket weighted options, $\xi(K,\alpha) = \left(K - \frac{1}{10}\sum_{j=1}^{10}(\alpha S^1_{t_j} + (1-\alpha)S^2_{t_j})\right)_+$,
$K \in [0.90, 1.10],\ \alpha \in [0,1]$.

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Kiitos huomiostanne!