Potentials of occupations measures of Gaussian fields and fractal minimizers

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Bibliography

Riesz kernel $k_\alpha(x) = c(n,\alpha)|x|^{\alpha-n}$, $x \in \mathbb{R}^n \setminus \{0\}$.

In the sequel: $c(n,\alpha) \equiv 1$.

Riesz potential for a nonnegative Radon measure μ,

$$U^\alpha \mu(x) := \int_{\mathbb{R}^n} k_\alpha(x-y)\,\mu(dy), \quad x \in \mathbb{R}^n.$$

Mutual Riesz energy for nonegative Radon measures μ and ν

$$I^\alpha(\mu,\nu) := \int_{\mathbb{R}^n} U^\alpha \mu(x)\,\nu(dx).$$

Riesz energy $I^\alpha(\mu) := I^\alpha(\mu,\mu)$.

Example: Coulomb interaction energy

Let $n \geq 3$, $\alpha = 2$, so $k_2(x) = |x|^{2-n}$.

Interpret finite discrete subsets $\mu = \{\mu_i\}$ and $\nu = \{\nu_i\}$ as configurations of positively charged particles.

$$I^2(\mu, \nu) = \sum_i \sum_j k_2(\mu_i - \nu_j)$$

is the *mutual Coulomb energy* of the discrete measures $\sum_i \delta_{\mu_i}$ and $\sum_j \delta_{\nu_j}$.

This is the part of the total electrostatic energy of the system caused by the *mutual Coulomb interaction* between μ and ν, self-interactions ignored.

If $\mu = \nu$, then $\sum_i \sum_j 1_{\{i \neq j\}} k_2(\mu_i - \mu_j)$ is the *internal electrostatic self-interaction energy* of a point configuration or a Coulomb gas.

Occupation measures

Let $X\colon [0,1]^k \to \mathbb{R}^n$ be a Borel function.

Occupation measure: $\mu_X(B) := \mathcal{L}^k(\{t \in [0,1]^k : X(t) \in B\})$, $B \subset \mathbb{R}^n$ Borel set.

Occupation time formula:

$$\int_{\mathbb{R}^n} g(x)\,\mu_X(dx) = \int_{[0,1]^k} g(X(t))\,dt, \quad \forall g \in \mathcal{B}_b(\mathbb{R}^n).$$

$X \mapsto \mu_X$ is highly nonlinear.

If $\mu_X \ll \mathcal{L}^n$, then

$$L_X := \frac{d\mu}{d\mathcal{L}^n} \in L^1(\mathbb{R}^n)$$

is called *local time* of X.

Notation

For $\gamma \in (0,1)$ and $X\colon [0,1]^k \to \mathbb{R}^n$, define $X \in \mathcal{C}_0^\gamma([0,1]^k;\mathbb{R}^n)$ if $X(0)=0$ and

$$\|X\|_{\mathcal{C}_0^\gamma} := \sup_{s\neq t} \frac{|X(t)-X(s)|}{|t-s|^\gamma} < \infty.$$

Set

$$\mathcal{B}_\varrho^\gamma := \{X \in C_0^\gamma([0,1]^k;\mathbb{R}^n) : \|X\|_{C_0^\gamma} \leq \varrho\}.$$

1.1 Hölder constrained minimal self-interaction

Recall that

$$X \mapsto I^\alpha(\mu_X) = \int_{[0,1]^k} \int_{[0,1]^k} |X(t) - X(s)|^{\alpha - n} ds\, dt.$$

Note that $\mu \mapsto I^\alpha(\mu)$ is quadratic and convex, but $X \mapsto I^\alpha(\mu_X)$ is not convex. Not even the effective domain $\{X\colon [0,1]^k \to \mathbb{R}^n :\ I^\alpha(\mu_X) < \infty\}$ is convex.

Theorem 1. *(Hinz, T., Viitasaari, 2025+) Let* $0<\alpha<n$, $0<\gamma<\frac{k}{n-\alpha}\wedge 1$.

i. *For any* $\varrho>0$*, there is some* $X^*\in\mathcal{B}_\varrho^\gamma$ *minimizing* $X\mapsto I^\alpha(\mu_X)$ *over* $\mathcal{B}_\varrho^\gamma$.

ii. *For any minimizer* X^* *of* $X\mapsto I^\alpha(\mu_X)$ *in* $\mathcal{B}_\varrho^\gamma$ *and any rectangle* $\mathcal{R}\subset[0,1]^k$*, the image* $X^*(\mathcal{R})$ *satisfies*

$$\mathcal{H}^{n-\alpha}(X^*(\mathcal{R}))=\infty,\quad \mathcal{H}^{\frac{k}{\gamma}\wedge n}(X^*(\mathcal{R}))<\infty$$

and, as a consequence,

$$n-\alpha\le\dim_H X^*(\mathcal{R})\le\frac{k}{\gamma}\wedge n.$$

Remark and Proof of (ii)

If $1 \le k \le n$, $\dim_H X^*(\mathcal{R})$ can be arbitrary close to $n-\alpha$, provided γ is close enough to $\frac{k}{n-\alpha} \wedge 1$. If α is non-integer, it forces $\dim_H X^*(\mathcal{R})$ to be non-integer.

Proof of (ii):

$\mathcal{H}^{n-\alpha}(X^*(\mathcal{R})) = \infty$ follows from $I^\alpha(\mu_{X^*}) < \infty$.

$\mathcal{H}^{\frac{k}{\gamma} \wedge n}(X^*(\mathcal{R})) < \infty$ follows from $X^* \in \mathcal{B}_\varrho^\gamma$.

(See Falconer [Fal90]).

For the proof of (i), we shall need two lemmas.

Lemma 2. *Every sequence* $(X_i)_i \in \mathcal{B}_\varrho^\gamma$ *has a subsequence* $(X_{i_j})_j$ *converging uniformly to some* $X^* \in \mathcal{B}_\varrho^\gamma$ *such that* $(\mu_{X_{i_j}})_j$ *converges weakly to* μ_{X^*}.

Proof of Lemma 2

Proof. The first statement follows from the Arzelà-Ascoli Theorem.

Since $\mathcal{L}^k([0,1]^k)=1$, the uniform convergence implies convergence in probability $\mathcal{L}^k|_{[0,1]^k}$. Let $\varphi\in C_c(\mathbb{R}^n)$, by uniform continuity, for every $\varepsilon>0$, there exists $\delta>0$ such that $|x-y|<\delta$ implies $|\varphi(x)-\varphi(y)|<\varepsilon$. By the occupation time formula

$$\left|\int_{\mathbb{R}^n}\varphi(x)\,\mu_{X_{i_j}}(dx)-\int_{\mathbb{R}^n}\varphi(x)\,\mu_{X^*}(dx)\right|\leq\int_{[0,1]^k}|\varphi(X_{i_j}(t))\,dt-\varphi(X^*(t))|\,dt$$

$$\leq\int_{|X_{i_j}-X^*|>\frac{\delta}{2}}|\varphi(X_{i_j}(t))\,dt-\varphi(X^*(t))|\,dt+\int_{|X_{i_j}-X^*|\leq\frac{\delta}{2}}|\varphi(X_{i_j}(t))\,dt-\varphi(X^*(t))|\,dt$$

$$\leq 2\|\varphi\|_\infty\mathcal{L}^k|_{[0,1]^k}\left(|X_{i_j}-X^*|>\frac{\delta}{2}\right)+\varepsilon\mathcal{L}^n(\mathrm{supp}(\varphi)).$$

By a density argument, $(\mu_{X_{i_j}})_j$ converges weakly to μ_{X^*}. □

Fractional Brownian fields

A centered Gaussian random field $b^H\colon [0,\infty)^k \times \Omega \to \mathbb{R}$ with $b^H(0) = 0$ $\mathbb{P}$-a.s. and

$$\mathbb{E}[|b^H(t) - b^H(s)|^2] = |t-s|^{2H}, \quad s,t \in [0,\infty)^k$$

over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is called a *fractional Brownian* $(k,1)$*-field* with *Hurst index* $H \in (0,1)$.

$k=1$: fractional Brownian motion (fBm)

$H = \frac{1}{2}$: Lévy's Brownian field

$k=1$, $H=\frac{1}{2}$: Brownian motion (Bm)

Note that for $\ell \in \mathbb{N}$, ℓ even

$$\mathbb{E}[|b^H(t) - b^H|^\ell] = (\ell-1)!!|t-s|^{\ell H}, \quad s,t \in [0,\infty)^k.$$

Fractional Brownian fields, cont'd

Let b_i^H, $1 \leq i \leq n$ be independent fractional Brownian $(k,1)$-fields with Hurst index $H \in (0,1)$. Then

$$B^H := (b_1^H, \dots, b_n^H) \colon [0,\infty)^k \times \Omega \to \mathbb{R}^n,$$

is a centered Gaussian field with $B^H(0) = 0$ $\mathbb{P}$-a.s. and

$$\mathbb{E}[|B^H(t) - B^H(s)|^\ell] = n^{\frac{\ell}{2}} \mathbb{E}[|b^H(t) - b^H(s)|^\ell], \quad s,t \in [0,\infty)^k$$

for $\ell \in \mathbb{N}$, ℓ even, called *fractional Brownian* (k,n)*-field* with *Hurst index* H.

Lemma 3

Lemma 3. *Let* $0<\alpha<n$, $0<\gamma<\frac{k}{n-\alpha}\wedge 1$, $\varrho>0$. *There exists* $X\in\mathcal{B}_\varrho^\gamma$ *such that* $I^\alpha(\mu_X)<\infty$.

Proof. By the Kolmogorov-Chentsov theorem there exists a modification of B^H (denoted by the same symbol) and a random variable $K>0$ and an event $\Omega_0\in\mathcal{F}$ with $\mathbb{P}(\Omega_0)=1$ such that

$$|B^H(t,\omega)-B^H(t,\omega)|\le K(\omega)|t-s|^\gamma,\quad s,t\in[0,1]^k,\quad \omega\in\Omega_0.$$

For some $c,c'>0$,

$$\mathbb{E}[|B^H(t)-B^H(s)|^{\alpha-n}]=c|t-s|^{-nH}\int_{\mathbb{R}^n}\exp\left(-\frac{|y|^2}{2|t-s|^{2H}}\right)|y|^{\alpha-n}dy$$

$$=c'|t-s|^{(\alpha-n)H}$$

Since $(n-\alpha)H<k$

$$\mathbb{E}\int_{[0,1]^k}\int_{\{s\in[0,1]^k:\,|B^H(t)-B^H(s)|<R\}}|B^H(t)-B^H(s)|^{\alpha-n}dtds$$

$$=c'\int_{[0,1]^k}\int_{\{s\in[0,1]^k:\,|B^H(t)-B^H(s)|<R\}}|t-s|^{(\alpha-n)H}dsdt<\infty$$

For $|z|\geq R$, $k_\alpha(z)\leq R^{\alpha-n}$, and hence there exists $\Omega_1\in\mathcal{F}$ with $\Omega_1\subset\Omega_0$ and $\mathbb{P}(\Omega_1)=1$ such that

$$I^\alpha(\mu_{B^H(\cdot,\omega)})<\infty \quad \forall\omega\in\Omega_1.$$

For $\omega\in\Omega_1$, set

$$X(t):=\varrho(1+K(\omega))^{-1}B^H(t,\omega), \quad t\in[0,1]^k,$$

then $X\in\mathcal{B}_\varrho^\gamma$ and $I^\alpha(\mu_X)<\infty$.

$\square$

Proof of Theorem 1. (The direct method in the calculus of variations).

Note that $\inf_{X \in \mathcal{B}_\varrho^\gamma} I^\alpha(\mu_X) < \infty$ by Lemma 3. If $(X_i)_i \subset \mathcal{B}_\varrho^\gamma$ is such that

$$\lim_{i\to\infty} I^\alpha(\mu_{X_i}) = \inf_{X \in \mathcal{B}_\varrho^\gamma} I^\alpha(\mu_X),$$

then, as $\mu \mapsto I^\alpha(\mu)$ is lower semi-continuous with respect to weak convergence, we have for X^* as in Lemma 1 that

$$\inf_{X \in \mathcal{B}_\varrho^\gamma} I^\alpha(\mu_X) \leq I^\alpha(\mu_{X^*}) \leq \liminf_{j\to\infty} I^\alpha(\mu_{X_{i_j}}) = \inf_{X \in \mathcal{B}_\varrho^\gamma} I^\alpha(\mu_X).$$

□

Remarks

- X^* is not expected to be unique. The radial symmetry of k_α produces easy counterexamples.
- For $K \subset \mathbb{R}^n$, K compact, $\mu \mapsto I^\alpha(\mu)$ admits a unique minimizer in the class $\mathcal{M}^1(K)$, called equilibrium measure μ_K of K with respect to I^α. For $K = \overline{B}(0, \varrho k^{\frac{\gamma}{2}})$, $I^\alpha(\mu_K) \leq I^\alpha(\mu_{X^*})$. However, μ_K is not an occupation measure in general, while μ_{X^*} has to satisfy $\operatorname{supp} \mu_{X^*} = X^*([0,1]^k)$.
- The **general philosophy** is that uniform convergence of the processes preserves **regularity**, whereas weak convergence of the occupation measures preserves **irregularity**.

Let ν be a compactly supported Borel probability measure on $\mathbb{R}^n$ (environment / medium). Consider

$$\min\left\{X \mapsto I^\alpha(\mu_X,\nu) = \int_{\mathbb{R}^n}\int_{[0,1]^k} |X(t)-y|^{\alpha-n} dt\,\nu(dy)\right\}$$

("mutual interaction energy" in "medium" ν, if $n=2$, $k=1$: "path in minefield").

Set

$$\mathcal{K} := \mathcal{K}(\gamma,\alpha,\varrho,M) := \left\{X \in \mathcal{B}_\varrho^\gamma : \sup_{x\in\operatorname{supp}\nu} U^\alpha\mu_X(x) \le M\right\}.$$

(If $k=1$: Hölder curves "whose velocity does not vary too wildly").

Remark. We need that $\operatorname{supp}\nu \cap \operatorname{supp}\mu_X \neq \emptyset$. Note that $\operatorname{supp}\mu_X \subset \overline{B}(0, \varrho k^{\frac{\gamma}{2}})$ is compact for $X \in \mathcal{B}_\varrho^\gamma$.

Doubly constrained minimal interaction

Theorem 4. *(Hinz, T., Viitasaari, 2025+) Let* $0 < \alpha < n$, $0 < \gamma < \frac{k}{n-\alpha} \wedge 1$, $\varrho > 0$, $M > 0$.

i. *If* $\mathcal{K} \neq \emptyset$, $\exists X^* \in \mathcal{K}$ *minimizing* $X \mapsto I^\alpha(\mu_X, \nu)$.

ii. *For any minimizer of* $X \mapsto I^\alpha(\mu_X, \nu)$ *and any rectangle* $\mathcal{R} \subset [0,1]^k$*,the image* $X^*(\mathcal{R})$ *satisfies*

$$\mathcal{H}^{n-\alpha}(X^*(\mathcal{R})) = \infty, \quad \mathcal{H}^{\frac{k}{\gamma} \wedge n}(X^*(\mathcal{R})) < \infty$$

and, as a consequence,

$$n - \alpha \leq \dim_H X^*(\mathcal{R}) \leq \frac{k}{\gamma} \wedge n.$$

iii. $\exists \varrho_1 > 0$ *and* $\exists M_1 > 0$ *depending on* γ, α, ν, *such that* $\mathcal{K} \neq \emptyset$, *whenever* $\varrho > \varrho_1$ *and* $M > M_1$.

Proof of Theorem 4. (Sketch).

i. As $y \mapsto U^\alpha \nu(y)$ is lower semi-continuous and nonnegative [Lan72], it follows that $X \mapsto I^\alpha(\mu_X, \nu)$ is lower semi-continuous in $L^1([0,1]^k; \mathbb{R}^n)$. Furthermore, it follows from properties of U^α that $\mathcal{K}$ is compact in L^1. The proof can be completed by the direct method in the calculus of variations, see [JL98].

ii. As before.

iii. We need to construct such an element in $\mathcal{K}$, see Lemma 6 below.

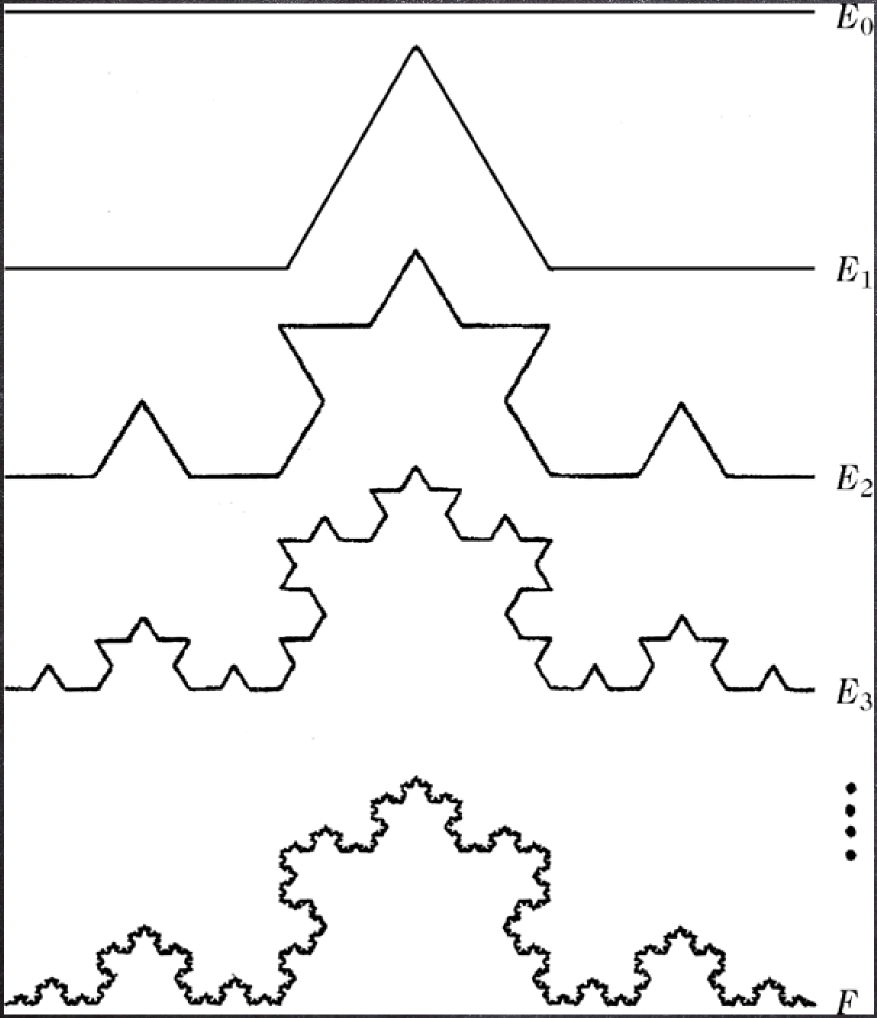
Alternative for (iii): Use Assouad's embedding theorem [Ass83] if

$$k\left(\lfloor \frac{1}{\gamma} \rfloor + 1\right) \leq n \leq \frac{k}{\gamma} + \alpha,$$

which implies that $\exists c, C > 0$ with

$$c|t-s|^\gamma \leq |X(t) - X(s)| \leq C|t-s|^\gamma.$$

Example: Koch curve $\gamma = \frac{\log 3}{\log 4}$.



Item (iii) of the previous proof has a probabilistic proof.

If $X\colon[0,1]^k \to \mathbb{R}^n$ has local times $L_X \in L^p_{\text{loc}}$ and $n-\alpha<\frac{n}{q}$ with $\frac{1}{p}+\frac{1}{q}=1$, then there exists $C>0$ with $U^\alpha\mu_X \le C$. If L_X is locally bounded, then $U^\alpha\mu_X$ is locally bounded for $0<\alpha<n$.

We claim that potentials of occupation measures are good replacements for L_X.

For technical reasons, replace $U^\alpha\mu_X$ by $G^\alpha\mu_X$, that is, G^α is the *Bessel potential*, i.e.

$$G^\alpha\mu(x)=\int_{\mathbb{R}^n} g_\alpha(x-y)\mu(dy), \quad x\in\mathbb{R}^n,$$

where the *Bessel kernel* g_α is defined via $\hat{g}_\alpha(\xi)=\left[(1+|\xi|^2)^{\frac{1}{2}}\right]^{-\alpha}$. U^α and G^α are comparable.

Local non-determinism

Let $X\colon [0,1]^k \times \Omega \to \mathbb{R}^n$ be a centered Gaussian random field, $X(0)=0$ $\mathbb{P}$-a.s. with $\sigma^2(s,t) := \operatorname{Cov}(X(t)-X(s))$.

We say that X is *locally non-deterministic* if $\forall \ell \geq 2$, $\exists c_\ell > 0$, $\exists \varepsilon_\ell > 0$ such that

$$\operatorname{Var}\left(\sum_{j=1}^{\ell} \langle u_j, \sigma_j^{-1}(X(t_j) - X(t_{j-1}))\rangle\right) \geq c_\ell \sum_{j=1}^{\ell} |u_j|^2,$$

for every $u_1, \dots, u_\ell \in \mathbb{R}^n$ and for a.e. $t_1, \dots, t_\ell \in ([0,1]^k)^\ell$ lying in a single subcube of side length ε_ℓ and such that $|t_{j+1} - t_j| \leq |t_{j+1} - t_i|$ for all $1 \leq i \leq j \leq \ell$. Here, $t_0 := 0$. Compare with [GH80].

Theorem 5. *(Hinz, T., Viitasaari, 2025+, for L_X, see [Pit78]) Let X be as before, and assume that X is locally non-deterministic. Let $\varphi\colon [0,1]^k \to \mathbb{R}^n$ be a bounded Borel function. Assume that $\det \sigma^2(s,t) > 0$. Let $0 < \alpha < n$, and assume that there exists $\delta > 0$ with*

$$\mathrm{esssup}_{s\in[0,1]^k} \int_{[0,1]^k} \frac{dt}{|\det \sigma^2(s,t)|^{\frac{n-\alpha}{2n}+\delta}} < \infty.$$

Then $G^\alpha \mu_{X+\varphi}$ is locally Hölder continuous.

Probabilistic proof of (iii) in Theorem 4

Lemma 6. *Let B^H be a fractional Brownian (k, n)-field with Hurst index $H \in (0, 1)$, let $\varphi\colon [0,1]^k \to \mathbb{R}^n$ be a bounded Borel function.*

i. If $n - \frac{k}{H} < \alpha < n$ and $0 < \gamma < H$, there exists $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$ such that $B^H(\cdot, \omega) \in C_0^\gamma([0,1]^k; \mathbb{R}^n)$ and $G^\alpha \mu_{B^H(\cdot,\omega)+\varphi} \in C(\mathbb{R}^n)$ as well as $G^\alpha \mu_{B^H(\cdot,\omega)+\varphi}$ and $U^\alpha \mu_{B^H(\cdot,\omega)+\varphi}$ are bounded for all $\omega \in \Omega_1$.

ii. If $n < \frac{k}{H}$ and $0 < \gamma < H$ there exists $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$ such that $B^H(\cdot, \omega) \in C_0^\gamma([0,1]^k; \mathbb{R}^n)$ and $B^H(\cdot, \omega) + \varphi$ has locally bounded local times $L_{B(\cdot,\omega)+\varphi} \in C(\mathbb{R}^n)$ for all $\omega \in \Omega_1$.

Proof. The hypotheses of Theorem 5 are satisfied, so we can apply the Kolmogorov-Chentsov theorem. □

Here, the situation is **typical**, in contrast to the Assouad embedding theorem.

Future topics

We have used **local non-determinism** to obtain **regularity** for $U^\alpha \mu_X$, $G^\alpha \mu_X$, which encodes the **irregularity** of X and implies certain **lower bounds** for X in terms of variation, see our preprint arXiv:2512.14248.

In the future, we would like to use potentials as replacements for L_X to obtain **lower bounds** for the paths. This is a publication in preparation with Hinz, T. and Viirtasaari.

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Thank you for your attention!