44th Finnish Summer School on Probability and Statistics

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Topics in Gaussian Wiener chaos expansion

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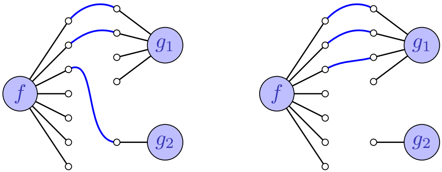

Plan

1. One-dimensional case
2. Multi-dimensional case
3. Gaussian fields
4. The Φ^4 model

Lecture notes: `arXiv/2605.14630`

Slides: `https://www.idpoisson.fr/berglund/Lammi26.pdf`

Some references:

- D. Nualart, *The Malliavin calculus and related topics,* Springer, 2006.
- G. Da Prato & L. Tubaro, *Wick powers in stochastic PDEs: an introduction.* 2007.
- M. Hairer, *Advanced stochastic calculus.* Lecture notes, EPFL & Imperial College London, 2026.
- NB, *An introduction to singular stochastic PDEs. Allen-Cahn equations, metastability, and regularity structures.* EMS Ser. Lect. Math., 2022.

1. The one-dimensional case

1. Gaussian random variables
2. Hermite polynomials
3. Wiener chaos expansion

Gaussian random variables

Definition: Gaussian random variable

$X \sim \mathcal{N}(m, \sigma^2)$ iff it has density

$$\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{e}^{-(x-m)^2/(2\sigma^2)}\,\mathrm{d}x$$

Properties:

1. $X \sim \mathcal{N}(m, \sigma^2) \Leftrightarrow X = m + \sigma Y$ with $Y \sim \mathcal{N}(0,1)$.
2. Assume $X \sim \mathcal{N}(m_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(m_2, \sigma_2^2)$ are defined on a common probability space, and let $Z = X + Y$. Then

$$Z \sim \mathcal{N}\big(m_1 + m_2, \sigma_1^2 + \sigma_2^2 + 2\,\mathrm{Cov}(X,Y)\big)$$

3. Two Gaussian variables X and Y are independent $\Leftrightarrow$ $\mathrm{Cov}(X,Y) := \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$.

Gaussian random variables

- **Aim:** If $X \sim \mathcal{N}(0,1)$, efficiently compute $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\mu(\mathrm{d}x)$
- **Example:** $\mathbb{E}[\mathrm{e}^{tX}] = \mathrm{e}^{t^2/2}$ (Laplace transform)

Proposition:

Let $X \sim \mathcal{N}(0,1)$. For any $n \in \mathbb{N}$, one has

$$\mathbb{E}[X^n] = \begin{cases} (n-1)!! & \text{if } n \text{ is even },\\ 0 & \text{if } n \text{ is odd }, \end{cases}$$

where

$$(n-1)!! = \prod_{k=0}^{n/2-1} (2k+1) = 1 \cdot 3 \cdot 5 \ldots (n-3)(n-1)$$

- Proofs:
  - Using Laplace transform
  - Using $\mathbb{E}[X^{n+1}] = n\mathbb{E}[X^{n-1}]$ (integration by parts)

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Hermite polynomials

1. **Linear algebra/geometry**: Gram–Schmidt
2. **Probability:** cumulants
3. **Analysis:** differential operators, spectral theory
4. **Algebra:** convolution algebra
5. **Combinatorics:** pairwise matchings

Gram–Schmidt orthogonalisation

- $(H_n(X))_{n\geqslant 0}$ orthogonal basis for $\langle X, Y\rangle = \mathbb{E}[XY]$ obtained from $(X^n)_{n\geqslant 0}$ by Gram–Schmidt

n	$H_n(x)$
0	1
1	x
2	$x^2 - 1$
3	$x^3 - 3x$
4	$x^4 - 6x^2 + 3$
5	$x^5 - 10x^3 + 15x$
6	$x^6 - 15x^4 + 45x^2 - 15$
7	$x^7 - 21x^5 + 105x^3 - 105x$
8	$x^8 - 28x^6 + 210x^4 - 420x^2 + 105$
9	$x^9 - 36x^7 + 378x^5 - 1260x^3 + 945$
10	$x^{10} - 45x^8 + 630x^6 - 3150x^4 + 472x^2 - 945$

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Cumulants

Definition: Moments and cumulants

- X r.v. such that $\mathbb{E}[e^{tX}] < \infty \, \forall t \in (-\delta, \delta)$

$$\mathbb{E}[e^{tX}] = \sum_{n \geqslant 0} \mu_n \frac{t^n}{n!}\,, \qquad \mu_n = \mathbb{E}[X^n] \quad \text{moments}$$

- Cumulant expansion of X:

$$K_X(t) = \log \mathbb{E}[e^{tX}] = \sum_{n \geqslant 0} \kappa_n \frac{t^n}{n!} \qquad \kappa_n : \text{ cumulants}$$

- $X \sim \mathcal{N}(0,1)$: $K_X(t) = \frac{t^2}{2}$, $\kappa_n = \delta_{n2}$
- $G(t,x) = \dfrac{e^{tx}}{\mathbb{E}[e^{tX}]} = e^{tx - t^2/2}$

Proposition: G is the generating function of the H_n

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Cumulants and Hermite polynomials

Proposition: Orthogonality

$$\mathbb{E}[H_n(X)H_m(X)] = n!\delta_{nm} = \begin{cases} n! & \text{if } n = m \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Proposition: Recurrence relation

$$H_{n+1}(x) = xH_n(x) - H_n'(x)$$

Proposition: Product–sum formula

$$H_n(x)H_m(x) = \sum_{p=0}^{n\wedge m} p!\binom{n}{p}\binom{m}{p} H_{n+m-2p}(x)$$

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Hermite polynomials and differential operators

▷ Define differential operators

$$a = \frac{\mathrm{d}}{\mathrm{d}x}\,, \qquad a^\dagger = x - \frac{\mathrm{d}}{\mathrm{d}x}\,, \qquad \mathscr{L} = -a^\dagger a = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x}$$

Proposition:

The operators a and $a^\dagger$ are mutually adjoint in $\mathscr{H} = L^2(\mathbb{R}, \mu(\mathrm{d}x))$, while $\mathscr{L}$ is self-adjoint and

$$aa^\dagger - a^\dagger a = \mathrm{id}$$

Corollary:

The Hermite polynomials are eigenfunctions of $\mathscr{L}$. More precisely,

$$(\mathscr{L}H_n)(x) = -nH_n(x) \qquad \forall n \geqslant 0$$

Furthermore,

$$a^\dagger H_{n-1} = H_n\,, \qquad aH_n = nH_{n-1} \qquad \forall n \geqslant 1$$

▷ $a^\dagger$ is called creation operator, a is called annihilation operator

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- Some consequences:

$$H_n'(x) = nH_{n-1}(x)$$
$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$$
$$H_n(x) = ((a^\dagger)^n H_0)(x) = (-1)^n \, \mathrm{e}^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x^2/2})$$

- $\mathscr{L}$ is infinitesimal generator of Ornstein–Uhlenbeck semigroup of SDE

$$\mathrm{d}x_t = -x_t \, \mathrm{d}t + \sqrt{2} \, \mathrm{d}W_t$$

- $H = \mathrm{e}^{-x^2/4} \mathscr{L} \, \mathrm{e}^{x^2/4}$ is Hamiltonian of quantum harmonic oscillator:

$$(Hf)(x) = \Big(\frac{1}{2} - \frac{x^2}{4}\Big) f(x) + f''(x)$$

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Convolution algebra

- $\mathbb{R}[x]$: algebra of polynomials in x, basis $(x^n)_{n\geqslant 0}$
- $\mathbb{R}[[t]]$: space of formal power series $\displaystyle\sum_{n\geqslant 0} \varphi_n \frac{t^n}{n!}$
- Let $\varphi : \mathbb{R}[x] \to \mathbb{R}$, and set $\varphi_n = \varphi(x^n)$. Define

$$\begin{aligned}\Lambda : \mathscr{L}(\mathbb{R}[x], \mathbb{R}) &\longrightarrow \mathbb{R}[[t]]\\ \varphi &\longmapsto \sum_{n\geqslant 0} \varphi(x^n) \frac{t^n}{n!}\end{aligned}$$

- Convolution product: $(\varphi * \psi)(x^n) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} \varphi(x^k)\psi(x^{n-k})$

Theorem:

Λ is an isomorphism between $\mathscr{L}(\mathbb{R}[x], \mathbb{R})$ and $\mathbb{R}[[t]]$

- $\varphi^{*p}(x^n) = \underbrace{(\varphi * \cdots * \varphi)}_{p \text{ factors}}(x^n) = \displaystyle\sum_{\substack{n_1,\ldots,n_p\geqslant 0\\ n_1+\cdots+n_p=n}} \frac{n!}{n_1!\ldots n_p!} \varphi(x^{n_1})\ldots\varphi(x^{n_p})$

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Operations on power series

- If $\varphi(1) = 1$ and $\psi(1) = 0$, set $\mathbf{1}^\star(x^n) = \delta_{n0}$ and define

$$\varphi^{-1} = \sum_{k\geqslant 0} (\mathbf{1}^\star - \varphi)^{*k}$$

$$\exp_*(\psi) = \sum_{k\geqslant 0} \frac{1}{k!}\psi^{*k} \qquad\qquad \log_*(\varphi) = \sum_{k\geqslant 1} \frac{(-1)^k}{k}(\varphi - \mathbf{1}^\star)^{*k}$$

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If $\varphi(1) = 1$ and $\psi(1) = 0$, then

$$\Lambda(\varphi^{-1})(t) = \left[\Lambda(\varphi)(t)\right]^{-1}$$

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- One has explicitly

$$\varphi^{-1}(x^n) = \sum_{k=1}^{n} (-1)^k \sum_{\substack{n_1,\dots,n_k\geqslant 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!}\varphi(x^{n_1})\dots\varphi(x^{n_k})$$

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Moments, cumulants and Wick map

- X real-valued random variable
- $\mu_X(x^n) = \mathbb{E}[X^n] \Rightarrow \Lambda(\mu_X)(t) = \mathbb{E}[e^{tX}]$
- Cumulant generating function:
  $K_X(t) = \log \mathbb{E}[e^{tX}] = \Lambda(\log_* \mu_X)(t) =: \Lambda(\kappa_X)(t) \qquad \mu_X = \exp_* \kappa_X$
- Leonov–Shiraev moment-cumulant relations:

$$\mu_X(x^n) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!} \kappa_X(x^{n_1})\dots\kappa_X(x^{n_k})$$

$$\kappa_X(x^n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1,\dots,n_k \geq 1 \\ n_1+\dots+n_k=n}} \frac{n!}{n_1!\dots n_k!} \mu_X(x^{n_1})\dots\mu_X(x^{n_k})$$

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  where $\Delta(x^n) := \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$ coproduct
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The Gaussian case

- For general X,

$$\mathrm{W}(x^n) = \sum_{k=0}^{n} \sum_{j=1}^{k} \frac{(-1)^j}{j!} \sum_{\substack{n_1,\dots,n_j \geqslant 1 \\ n_1+\dots+n_j=k}} \frac{n!}{(n-k)!n_1!\dots n_j!} \kappa_X(x^{n_1}) \dots \kappa_X(x^{n_j}) x^{n-k}$$

- For $X \sim \mathcal{N}(0,1)$, $\kappa_X(x^n) = \delta_{n2}$,

$$\mathbb{E}[X^{2k}] = \mu_X(x^{2k}) = \frac{(2k)!}{k!2^k} = (k-1)!!$$

Proposition: Explicit expressions for Hermite polynomials

For all $n \in \mathbb{N}_0$,

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{2^k k!(n-2k)!} x^{n-2k}$$

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Hermite polynomials and combinatorics

Theorem:

Let $E_n = [\![1, n]\!] := \{1, 2, \ldots, n\}$ and let $0 \leqslant 2k \leqslant n$. The coefficient of x^{n-2k} of $H_n(x)$ is equal to the number of pairwise matchings of E_n with k pairs

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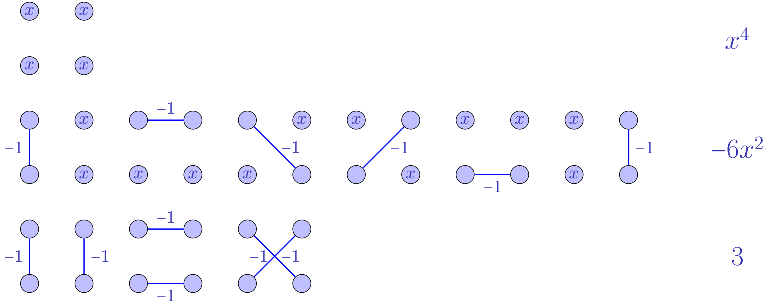

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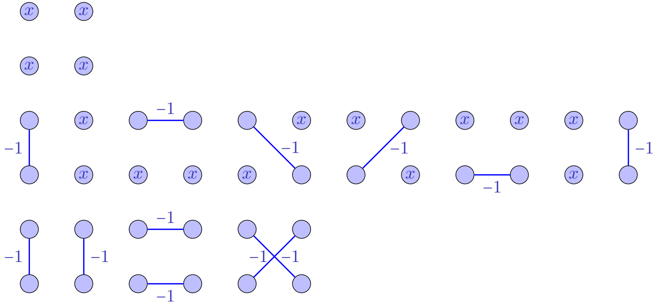

x^4

$-6x^2$

3

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Wiener chaos decomposition

Lemma:

The r.v. $\{e^{tX} : t \in \mathbb{R}\}$ form a total subset of $\mathscr{H} = L^2(\mathbb{R}, \mu(dx))$

Definition: Wiener chaos

For any $n \geqslant 1$, let $\mathscr{H}_n$ be the one-dimensional subspace of $\mathscr{H}$ spanned by the random variable $H_n(X)$. For $n = 0$, $\mathscr{H}_0$ is the set of constants, isomorphic $\mathbb{R}$. Then $\mathscr{H}_n$ is called the homogeneous Wiener chaos of order n. The inhomogeneous Wiener chaos of order n is

$$\mathscr{H}_{\leqslant n} = \bigoplus_{k=0}^{n} \mathscr{H}_k$$

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2. The multi-dimensional case

1. Wick calculus
2. Hermite polynomials for multivariate Gaussians
3. Wiener chaos expansion
4. Equivalence of moments

Multivariate Gaussian random variables

Definition: Multivariate Gaussian

For $N \geqslant 1$, let $\mathbb{R}^N$ be equipped with the σ-algebra $\mathcal{B}$ of Borel sets and Lebesgue measure $\mathrm{d}x$. Let $m \in \mathbb{R}^N$ and let $C \in \mathbb{R}^{N\times N}$ be a symmetric, positive definite matrix. A r.v. $X : \mathbb{R}^n \to \mathbb{R}$ is a (multivariate) Gaussian random variable with mean m and covariance matrix C if its law is

$$\mu(\mathrm{d}x) = \frac{1}{(2\pi)^{N/2} \det(C)^{1/2}} \, \mathrm{e}^{-\langle (x-m), C^{-1}(x-m)\rangle/2} \, \mathrm{d}x$$

In that case, we write $X \sim \mathcal{N}(m, C)$.

Proposition: Laplace transform

For C sym. pos. def., $X \sim \mathcal{N}(0, C) \Leftrightarrow \mathbb{E}[\mathrm{e}^{\langle t, X\rangle}] = \mathrm{e}^{\langle t, Ct\rangle/2} \ \forall t \in \mathbb{R}^N$

Corollary: Covariance

If $X \sim \mathcal{N}(0, C)$, then $\mathbb{E}[X_i X_j] = C_{ij}$ for all $i, j \in [\![1, N]\!]$

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Isserlis' theorem

Lemma: Integration by parts

Assume $X \sim \mathcal{N}(0, C)$. For any $i \in [\![1, N]\!]$ and differentiable $f : \mathbb{R}^N \to \mathbb{R}$,

$$\mathbb{E}\big[X_i f(X)\big] = \sum_{j=1}^{N} C_{ij} \mathbb{E}\big[\partial_j f(X)\big]$$

Theorem: [Isserlis]

For $1 \leqslant k \leqslant \frac{N}{2}$, $\mathbb{E}\big[X_1 \dots X_{2k-1}\big] = 0$ and

$$\mathbb{E}\big[X_1 \dots X_{2k}\big] = \sum_{\mathscr{P}} \prod_{\{i,j\} \in \mathscr{P}} \mathbb{E}\big[X_i X_j\big]$$

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$$\mathbb{E}\big[X_1X_2X_3X_4\big] = \mathbb{E}\big[X_1X_2\big]\mathbb{E}\big[X_3X_4\big] + \mathbb{E}\big[X_1X_3\big]\mathbb{E}\big[X_2X_4\big] + \mathbb{E}\big[X_1X_4\big]\mathbb{E}\big[X_2X_3\big]$$

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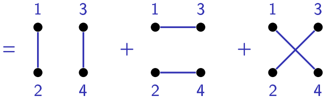

Scaled Hermite polynomials

Definition: Scaled Hermite polynomials

The Hermite polynomial of degree n with variance σ^2 is defined as

$$H_n(x;\sigma^2) = \sigma^n H_n(x/\sigma)$$

- Generating function: $G(t,x) = e^{tx-\sigma^2 t^2/2}$
- Recursive relations:

$$H_{n+1}(x;\sigma^2) = xH_n(x;\sigma^2) - \sigma^2 \partial_x H_n(x;\sigma^2)$$
$$\partial_x H_n(x;\sigma^2) = nH_{n-1}(x;\sigma^2)$$

- Explicit expression:

$$H_n(x;\sigma) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{2^k k!(n-2k)!} \sigma^{2k} x^{n-2k}$$

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Binomial formula

Lemma: Binomial formula

For any $x, y \in \mathbb{R}$, any $\sigma_1, \sigma_2 \in \mathbb{R}$ and any $n \in \mathbb{N}_0$, one has

$$H_n(x+y;\sigma_1^2+\sigma_2^2) = \sum_{m=0}^{n} \binom{n}{m} H_m(x;\sigma_1^2) H_{n-m}(y;\sigma_2^2)$$

Proposition: Multinomial formula

Let $a \in \ell^2$ be a sequence of real numbers such that $\sum_{i\geqslant 0} a_i^2 = 1$. Then for any sequence $(x_i)_{i\geqslant 0}$ such that $\sum_{i\geqslant 0} a_i x_i$ converges, one has

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$$k! := \prod_{i\geqslant 0} k_i! \,, \qquad a^k := \prod_{i\geqslant 0} a_i^{k_i}$$

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where the sum runs over all $k \in \mathbb{N}_0^{\mathbb{N}_0}$ such that $|k| = \sum_{i\geqslant 0} k_i = n$, and

$$k! := \prod_{i\geqslant 0} k_i! \,, \qquad a^k := \prod_{i\geqslant 0} a_i^{k_i}$$

Wiener chaos expansion

- $X_1, \dots, X_N$ iid $\mathcal{N}(0,1)$ on $(\Omega, \mathscr{F}, \mathbb{P})$, $\mathscr{H} = L^2(\Omega, \mathscr{F}, \mathbb{P})$.
- $\mathbf{H} = \mathbb{R}^N$. Define $W : \mathbf{H} \to \mathscr{H}$ by $W(h) = \sum_{i=1}^{N} h_i X_i$.

Definition: Wiener chaos

For any $n \geqslant 1$, let $\mathscr{H}_n$ be the subspace of $\mathscr{H}$ spanned by the r.v.

$$\{H_n(W(h)) : h \in \mathbf{H}, \|h\|_{\mathbf{H}} = 1\}$$

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Wiener isometry

- For $k \in \mathbb{N}_0^N$, $\Phi_k := \prod_{i=1}^{N} H_{k_i}(X_i)$. Orthogonality: $\mathbb{E}[\Phi_k \Phi_\ell] = k!\delta_{k\ell}$.
- $\mathbf{H}^{\otimes_s n}$: symmetric tensors in $\mathbf{H}^{\otimes n}$.
- Projection: $\Pi\big(h_1 \otimes \cdots \otimes h_n\big) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}$.
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For $k \in \mathbb{N}_0^N$, let $n = |k|$. The nth Wiener isometry is the map $\mathbf{H}^{\otimes_s n} \to \mathscr{H}_n$

$$I_n : e_k \longmapsto \frac{1}{\sqrt{n!}} \Phi_k$$

- $I_0 = 1$, $I_1(h) = W(h)$.

Lemma:

If $\|h\|_{\mathbf{H}} = 1$, then $I_n(h^{\otimes n}) = \frac{1}{\sqrt{n!}} H_n\big(W(h)\big)$

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Multiplication

- New normalisation: $\hat{I}_n(f) = \sqrt{n!} I_n(f)$.
- Notation: $f = \sum_{i=1}^N f(i) e_i \in \mathbf{H}$.

$$\mathbf{H}^{\otimes n} \ni f_1 \otimes \cdots \otimes f_n = \sum_{i_1, \ldots, i_n = 1}^{N} \underbrace{f_1(i_1) \ldots f_n(i_n)}_{=: f(i_1, \ldots, i_n)} e_{i_1} \otimes \cdots \otimes e_{i_n}$$

- Shuffles: $\mathfrak{S}(p, n) \subset \mathfrak{S}(n)$ permutations of $[\![1, n]\!]$ preserving order of p first and $n - p$ last elements.

Lemma:

Assume $f \in \mathbf{H}^{\otimes n}$ and $g \in \mathbf{H}$. Then

$$\hat{I}_n(f) \hat{I}_1(g) = \hat{I}_{n+1}(f \otimes g) + \hat{I}_{n-1}(f \star_1 g)$$

where $\star_1$ denotes the contraction operation

$$(f \star_1 g)(i_1, \ldots, i_{n-1}) = \sum_{\Sigma \in \mathfrak{S}(1,n)} \sum_{j=1}^{N} f(\Sigma(j, i_1, \ldots, i_{n-1})) g(j)$$

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Multiplication

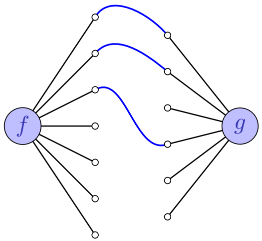
Proposition: Multiplication between nth and mth chaos

Assume $f \in \mathbf{H}^{\otimes n}$ and $g \in \mathbf{H}^{\otimes m}$. Then

$$\hat{I}_n(f)\hat{I}_m(g) = \sum_{p=0}^{n\wedge m} \hat{I}_{n+m-2p}(f \star_p g)$$

where $\star_0 = \otimes$ and for $\mathbf{i} = (i_1, \ldots, i_{n-p})$ and $\mathbf{j} = (j_1, \ldots, j_{m-p})$

$$(f \star_p g)(\mathbf{i},\mathbf{j}) = \sum_{\substack{\Sigma \in \mathfrak{S}(p,n) \\ \bar{\Sigma} \in \mathfrak{S}(p,m)}} \sum_{\sigma \in \mathfrak{S}(p)} \sum_{\mathbf{k} \in [[1,N]]^p} f(\Sigma(\mathbf{k},\mathbf{i}))g(\bar{\Sigma}(\mathbf{k},\sigma(\mathbf{j})))$$



Equivalence of moments

Theorem: Equivalence of moments

Assume F belongs to the nth Wiener chaos $\mathscr{H}_n$. Then for any $p > 1$,

$$\mathbb{E}[F^{2p}]^{1/2p} \leqslant (2p-1)^{n/2} \mathbb{E}[F^2]^{1/2}$$

Definition: Ornstein–Uhlenbeck semigroup

The Ornstein–Uhlenbeck semigroup is the one-parameter semigroup $\{T_t : t \geqslant 0\}$ of contraction operators on $\mathscr{H}$ defined by

$$T_t(F) = \sum_{n=0}^{\infty} \mathrm{e}^{-nt} P_n F$$

for any $F \in \mathscr{H}$, where $P_n : \mathscr{H} \to \mathscr{H}_n$ denotes the orthogonal projection on the nth Wiener chaos.

- OU process: $\mathrm{d}X_t = -X_t\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t$, $X_0 = x$.
- $\mathbb{E}[H_n(X_t)] = H_n(x)\,\mathrm{e}^{-nt} \Rightarrow \mathbb{E}[f(X_t)] = T_t(f)(x)$.

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Hypercontractivity

Proposition: Mehler's formula

Let $W' = \{W'(h) : h \in \mathbf{H}\}$ be an independent copy of $W = \{W(h) : h \in \mathbf{H}\}$, where W and W' are defined on a product space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \times \mathbb{P}')$. For $t > 0$, consider the process $Z = \{Z(h) : h \in \mathbf{H}\}$, defined by

$$Z(h) = \mathrm{e}^{-t} W(h) + \sqrt{1 - \mathrm{e}^{-2t}} W'(h)$$

Then for any $F \in \mathscr{H}$ of the form $F = f(W)$, one has

$$T_t(F) = \mathbb{E}'\left[f(Z)\right]$$

where $\mathbb{E}'$ denotes the expectation with respect to the law $\mathbb{P}'$ of W'.

Theorem: Hypercontractivity of the OU semigroup

For $p > 1$ and $t > 0$, let

$$q(t) = \mathrm{e}^{2t}(p - 1) + 1 > p$$

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3. Gaussian fields

1. Isonormal Gaussian processes
2. Gaussian white noise
3. The Gaussian free field

Isonormal Gaussian processes

Definition: Isonormal Gaussian process

Let $\mathbf{H}$ be a separable Hilbert space. A stoch. process $W = \{W(h) : h \in \mathbf{H}\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an isonormal Gaussian process if W is a centred Gaussian family of random variables such that

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$$\mathbf{H}^{\otimes n} \ni h = h_1 \otimes \cdots \otimes h_n = \sum_{i_1 \geqslant 0, \ldots, i_n \geqslant 0} \underbrace{\hat{h}_1(i_1) \ldots \hat{h}_n(i_n)}_{= \hat{h}(i_1, \ldots, i_n)} e_{i_1} \otimes \cdots \otimes e_{i_n}$$

$$h(x_1, \ldots, x_n) = \sum_{i_1 \geqslant 0, \ldots, i_n \geqslant 0} \hat{h}(i_1, \ldots, i_n) e_{i_1}(x_{i_1}) \ldots e_{i_n}(x_{i_n})$$

Lemma:

Let $f \in \mathbf{H}^{\otimes n}$ and $g \in \mathbf{H}^{\otimes m}$. For any $p \leqslant n \wedge m$, all $x \in \Lambda^{n-p}$ and $y \in \Lambda^{m-p}$,

$$(f \star_p g)(x, y) = \sum_{\substack{\Sigma \in \mathfrak{S}(p,n) \\ \bar{\Sigma} \in \mathfrak{S}(p,m)}} \sum_{\sigma \in \mathfrak{S}(p)} \int_{\Lambda^p} f\big(\Sigma(z, x)\big) g\big(\bar{\Sigma}(z, \sigma(x))\big) \, \mathrm{d}z$$

Gaussian fields

- For $h = \sum_{i\geqslant 0} \hat{h}(i) e_i \in \mathbf{H}$ set

$$\Psi(h) = \sum_{i\geqslant 0} \hat{h}(i) W(e_i) e_i = \sum_{i\geqslant 0} \hat{h}(i) X_i e_i$$

- Then $\Psi(h)(x) = \sum_{i\geqslant 0} \hat{h}(i) X_i e_i(x)$ is a random field.
- $\|\Psi(h)\|_{\mathbf{H}}^2 = \sum_{i\geqslant 0} \hat{h}(i)^2 X_i^2$.
- $\mathbb{E}\big[\|\Psi(h)\|_{\mathbf{H}}^2\big] = \|h\|_{\mathbf{H}}^2$.
- Ψ is an isometry from $\mathbf{H}$ to $\widetilde{\mathscr{H}_1} \subset \widetilde{\mathscr{H}}$, space of $\mathbf{H}$-valued random variables with finite variance.
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Gaussian white noise

- $\hat{h} = (1,1,1,\dots) \notin \mathbf{H}$
  $$\Rightarrow \quad \xi(x) := \Psi(h)(x) = \sum_{i\geqslant 0} X_i e_i(x) \text{ is called white noise on } \Lambda.$$
- Mollification with cut-off N: $\hat{h}_N = (\underbrace{1,1,1,\dots,1}_{N \text{ components}},0,0,\dots) \in \mathbf{H}$
  $$\Rightarrow \quad \xi_N(x) := \Psi(h_N)(x) = \sum_{i=0}^{N} X_i e_i(x).$$
- $\varphi : \Lambda \to \mathbb{R}$ test function.
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Definition: Gaussian white noise on the torus

Gaussian white noise on $\mathbb{T}^d$ is the random distribution ξ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any smooth test functions $\varphi, \varphi_1, \varphi_2 \in \mathbf{H}$, $\langle \xi, \varphi\rangle \sim \mathcal{N}(0, \|\varphi\|_{\mathbf{H}}^2)$ and
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Properties of Gaussian white noise

- Scaling: $(\mathscr{S}^\lambda \varphi)(x) = \frac{1}{\lambda^d}\varphi\left(\frac{x}{\lambda}\right)$

Lemma: Scaling of white noise

Let $\langle \xi_\lambda, \varphi\rangle = \langle \xi, \mathscr{S}^\lambda \varphi\rangle$. For any $\lambda \in (0,1]$, one has $\xi_\lambda \overset{\text{law}}{=} \frac{1}{\lambda^{d/2}}\xi$

- Fourier basis: $e_k(x) = \mathrm{e}^{2\pi\,\mathrm{i}\langle k,x\rangle}$. Covariance: $\mathbb{E}[X_k X_\ell] = \delta_{k,-\ell}$.
  $(\mathrm{id}-\Delta)e_k(x) = \lambda_k e_k(x)$ where $\lambda_k = 1 + (2\pi)^d\|k\|^2$.

Definition: Fractional Sobolev spaces

For $s \geqslant 0$, $H^s(\Lambda) = \{f \in \mathbf{H} \colon \|f\|_{H^s} < \infty\}$, where

$$\|f\|_{H^s}^2 := \sum_{k\in\mathbb{Z}^d} \lambda_k^s |\hat{f}(k)|^2 < \infty$$

For $s < 0$, $H^s(\Lambda)$ is the closure of $L^2(\Lambda)$ under the norm $\|\cdot\|_{H^s}$

Proposition: Sobolev regularity of white noise on the torus

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- B_r : set of smooth test functions $\varphi : \Lambda \to \mathbb{R}$, supported on a ball or radius 1, whose partial derivatives up to order r are bounded by 1.

Definition: Hölder–Besov spaces

For $\alpha < 0$, the space $\mathscr{C}^\alpha(\Lambda)$ consists in all Schwartz distributions $\zeta \in \mathcal{S}'(\Lambda)$ such that

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where $r = \lceil -\alpha \rceil$.

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White noise ξ belongs to $\mathscr{C}^\alpha$ for any $\alpha < -\frac{d}{2}$

- **Remark:** $H^s = \mathscr{B}^s_{2,2}$ and $\mathscr{C}^\alpha = \mathscr{B}^\alpha_{\infty,\infty}$, where $\mathscr{B}^\alpha_{p,q}$ are Besov spaces.

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The Gaussian free field

- $\hat{h}(k) = \frac{1}{\sqrt{\lambda_k}}$, $\lambda_k = 1 + (2\pi)^d \|k\|^2$, $k \in \mathbb{Z}^d$
$$\Rightarrow \quad \phi_{\mathrm{GFF}}(x) := \Psi(h)(x) = \sum_{k \in \mathbb{Z}^d} \frac{X_k}{\sqrt{\lambda_k}} e_k(x)$$
- $\|h\|_{\mathbf{H}}^2 = \sum_{k \in \mathbb{Z}^d} \frac{1}{\lambda_k} < \infty \Leftrightarrow d < 2.$
- Covariance: $\mathbb{E}\big[\phi_{\mathrm{GFF}}(x)\phi_{\mathrm{GFF}}(y)\big] = \sum_{k \in \mathbb{Z}^d} \frac{e_k(x-y)}{\lambda_k} =: G(x-y).$

Lemma:

For any $g \in \mathbf{H}$, the function f defined by $f(x) = \int_\Lambda G(x-y) g(y)\,\mathrm{d}y$ satisfies $(\mathrm{id} - \Delta) f(x) = g(x)$.

Definition: Green function and GFF

- $G = (\mathrm{id} - \Delta)^{-1}$ is the Green function of $\mathrm{id} - \Delta$.
- ϕ_{GFF} is the Gaussian free field (GFF) of covariance $(\mathrm{id} - \Delta)^{-1}$.

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The Gaussian free field on $\mathbb{T}^1$

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For $0 < \alpha < 1$, the space $\mathscr{C}^\alpha(\Lambda)$ consists in all functions $f : \Lambda \to \mathbb{R}$ such that

$$\|f\|_{\mathscr{C}^\alpha} = \sup_{x\in\Lambda}|f(x)| + \sup_{\substack{x,y\in\Lambda\\ x\neq y}} \frac{|f(x) - f(y)|}{\|x-y\|^\alpha} < \infty$$

Proposition: Hölder–Besov regularity of the GFF on $\mathbb{T}^1$

The GFF on the circle belongs to $\mathscr{C}^\alpha$ for any $\alpha < \frac{1}{2}$

- Proof uses Kolmogorov's continuity criterion: $\mathbb{E}\big[\|\phi(y) - \phi(x)\|^\mu\big] \leqslant C|y-x|^{1+\nu}\ \forall x,y \Rightarrow \phi \in \mathscr{C}^\alpha\ \forall\alpha < \frac{\nu}{\mu}$.

Proposition: Moments of the GFF on $\mathbb{T}^1$

For any $p > 1$, there exists a constant $C(p)$ such that

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The Gaussian free field on $\mathbb{T}^1$

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The Gaussian free field on $\mathbb{T}^2$

Definition: Truncated two-dimensional Gaussian free field

For $N \geqslant 1$, let $\mathcal{K}_N = \{k \in \mathbb{Z}^2 \colon |k| \leqslant N\}$, where $|k| = |k_1| + |k_2|$. The truncated GFF with covariance $(\mathrm{id} - \Delta_N)^{-1}$ on Λ is defined as

$$\phi_{\mathrm{GFF},N}(x) := \sum_{k \in \mathcal{K}_N} \frac{X_k}{\sqrt{\lambda_k}} e_k(x)$$

Here Δ_N is the restriction of Δ to the subspace E_N of $\mathbf{H}$ spanned by Fourier basis functions e_k with $|k| \leqslant N$.

- $\mathbb{E}[\phi_{\mathrm{GFF},N}(x)\phi_{\mathrm{GFF},N}(y)] = \sum_{k \in \mathcal{K}_N} \frac{1}{\lambda_k} e_k(x-y) =: G_N(x-y)$.
- $C_N = G_N(0) \asymp \log(N)$.

Proposition: Uniform bound on the variance of Wick powers

$$\sup_{N \geqslant 1} \mathbb{E}\left[\left(\int_\Lambda :\phi_{\mathrm{GFF},N}^n(x): \mathrm{d}x\right)^{2p}\right] < \infty \qquad \forall n \geqslant 1, p \geqslant 1$$

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4. The Φ^4 model

1. The Φ^4_1 model
2. The Φ^4_2 model
3. The Φ^4_3 model
4. The $\Phi^4_{4-\varepsilon}$ model

The Φ^4_d model

- $\Lambda = \mathbb{T}^d$, $\phi : \Lambda \to \mathbb{R}$, $\alpha \geqslant 0$, $m > 0$.

  Energy: $\mathcal{H}_{d,\alpha}(\phi) = \int_\Lambda \Big[\|\nabla\phi(x)\|^2 + \frac{m^2}{2}\phi(x)^2 + \alpha\phi(x)^4 \Big] \mathrm{d}x$

- **Aim:** compute expectations under Gibbs measure

  $\mu_{d,\alpha} \sim \frac{1}{\mathscr{Z}_{d,\alpha}} \mathrm{e}^{-\mathcal{H}_{d,\alpha}(\phi)} \, \mathrm{d}\phi$ $\qquad$ $\mathscr{Z}_{d,\alpha}$: partition function

- Case $\alpha = 0$, $m = 1$: $\mathcal{H}_{d,0}(\phi) = \frac{1}{2}\langle \phi, [\mathrm{id} - \Delta]\phi \rangle_{\mathbf{H}}$

  $$\Rightarrow \mathbb{E}^{\mu_{d,0}}[F] = \mathbb{E}\Big[F\Big(\sum_{k\in\mathbb{Z}^d} \frac{X_k}{\sqrt{\lambda_k}} e_k \Big) \Big]$$

  Example: $\mathbb{E}^{\mu_{d,0}}[\phi(x)\phi(y)] = G(x-y)$

- Case $\alpha > 0$:

  $$\mathbb{E}^{\mu_{d,\alpha}}[F] = \frac{\mathscr{Z}_{d,0}}{\mathscr{Z}_{d,\alpha}} \mathbb{E}^{\mu_{d,0}}\Big[F(\phi) \exp\Big\{ -\alpha \int_\Lambda \phi(x)^4 \, \mathrm{d}x \Big\} \Big]$$

  In particular, $F = 1 \Rightarrow \frac{\mathscr{Z}_{d,\alpha}}{\mathscr{Z}_{d,0}} = \mathbb{E}^{\mu_{d,0}}\Big[\exp\Big\{ -\alpha \int_\Lambda \phi(x)^4 \, \mathrm{d}x \Big\} \Big]$

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The Φ_1^4 model – Feynman diagrams

- Consider $\mathcal{H}^{\text{Wick}}_{1,\alpha}(\phi) = \int_\Lambda \Big[\|\nabla\phi(x)\|^2 + \frac{1}{2}\phi(x)^2 + \alpha :\phi(x)^4: \Big] \mathrm{d}x$

  where $:\phi(x)^4: = H_4(\phi(x)^4; C)$, with $C = G(0)$ (choice of m).

- To be computed: $\dfrac{\mathscr{Z}_{1,\alpha}}{\mathscr{Z}_{1,0}} \asymp \sum_{n\geqslant 0} \dfrac{(-\alpha)^n}{n!} \mathbb{E}^{\mu_{1,0}}\Big[\Big(\int_\Lambda :\phi(x)^4: \mathrm{d}x \Big)^n \Big]$

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  Notation: $\int_\Lambda \int_\Lambda G(x-y)^4 \,\mathrm{d}x\,\mathrm{d}y =: \Pi\Big($ $\Big)$

- **Remark:** If $h_x := \sum_{k\in\mathbb{Z}} \hat{h}_x(k) e_k$ where $\hat{h}_x(k) := \dfrac{e_k(x)}{\sqrt{\lambda_k}}$

  then $\hat{I}_1(h_x) = \phi(x)$ and $:\phi(x)^4: = H_4(\phi(x); C) = \hat{I}_4(h_x^{\otimes 4})$

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The Φ^4_1 model – Feynman diagrams

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The Φ^4_1 model – Feynman diagrams

- Term $n = 3$:

$$
\begin{aligned}
:\phi(x)^4::\phi(y)^4::\phi(z)^4: &= \hat{I}_4(h_x^{\otimes 4})\hat{I}_4(h_y^{\otimes 4})\hat{I}_4(h_z^{\otimes 4}) \\
&= \sum_{p=0}^{4} \hat{I}_{8-2p}(h_x^{\otimes 4} \star_p h_y^{\otimes 4})\hat{I}_4(h_z^{\otimes 4}) \\
&= \sum_{p=0}^{4} \sum_{q=0}^{(8-2p)\wedge 4} \hat{I}_{12-2p-2q}\Big((h_x^{\otimes 4} \star_p h_y^{\otimes 4}) \star_q h_z^{\otimes 4}\Big) .
\end{aligned}
$$

$$
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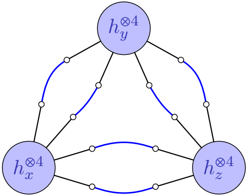

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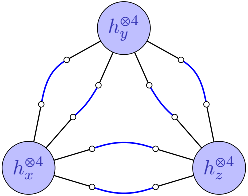

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The Φ^4_1 model – Feynman diagrams

Definition: Vacuum Feynman diagram

A vacuum diagram is a multigraph $\Gamma = (\mathscr{V}, \mathscr{E})$, meaning there can be multiple edges between vertices. Its valuation is defined by

$$\Pi(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e \in \mathscr{E}} G(x_{e_+} - x_{e_-})\, \mathrm{d}x$$

where $e_\pm$ are the vertices connected by the edge e.

Example: $\int_{\Lambda^3} G(x-y)^2 G(y-z)^2 G(x-y)^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \Pi\Big($ $\Big)$

Proposition: Expansion of moments into Feynman diagrams

For any $n \geqslant 2$,

$$\mathbb{E}^{\mu_{1,0}}\Big[\Big(\int_\Lambda :\phi(x)^4: \mathrm{d}x\Big)^n\Big] = \sum_k \Pi(\Gamma_{n,k})$$

where the sum runs over all vacuum diagrams $\Gamma_{n,k}$ with n vertices and $2n$ edges, obtained as perfect pairwise matchings of n vertices of arity 4.

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The linked-cluster theorem

Proposition: Linked-cluster theorem

The cumulant expansion of the ratio of partition functions is given by

$$\log \frac{\mathscr{Z}_{1,\alpha}}{\mathscr{Z}_{1,0}} \asymp \sum_{n \geqslant 0} \frac{(-\alpha)^n}{n!} \sum_{k:\Gamma_{n,k}\ \text{connected}} \Pi(\Gamma_{n,k})$$

- **Example:** $\psi(x) = 0 \Rightarrow \exp_*(\psi)(x^4) = \psi(x^4) + \binom{4}{2}\psi(x^2)^2$

$$\Rightarrow \quad \mathbb{E}^{\mu_{1,0}}\Big[\exp\Big(-\alpha \int_\Lambda :\phi(x)^4: \mathrm{d}x\Big)^4\Big]$$

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Asymptotic series

Proposition: Asymptotic series

For every $n \geqslant 0$ there exists a constant M_n such that the ratio of partition functions satisfies

$$\left| \frac{\mathscr{Z}_{1,\alpha}}{\mathscr{Z}_{1,0}} - \sum_{m=0}^{n} \frac{(-\alpha)^m}{m!} \mathbb{E}^{\mu_{1,0}} \left[\left(\int_\Lambda :\phi(x)^4: \mathrm{d}x \right)^m \right] \right| \leqslant M_n \alpha^{n+1}$$

Notation: $\mathbf{X} := \int_\Lambda :\phi(x)^4: \mathrm{d}x$.

Lemma:

There exists $\alpha_0 > 0$ such that for all $\alpha \in [0, \alpha_0)$, one has

$$0 \leqslant \mathbb{E}^{\mu_{1,0}} \left[\mathrm{e}^{-\alpha \mathbf{X}} \right] \leqslant 1 + \mathcal{O}(\alpha)$$

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The two-point function

- ▷ $G_{2,1,\alpha}(x,y) = \mathbb{E}^{\mu_{1,\alpha}}[\phi(x)\phi(y)] = \dfrac{\mathscr{Z}_{1,0}}{\mathscr{Z}_{1,\alpha}}\mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)\,\mathrm{e}^{-\alpha\mathbf{X}}]$
- ▷ $\mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)\,\mathrm{e}^{-\alpha\mathbf{X}}] \asymp \displaystyle\sum_{n\geqslant 0}\frac{(-\alpha)^n}{n!}\mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)\mathbf{X}^n]$
- ▷ Term $n=0$: $\mathbb{E}^{\mu_{1,0}}[\phi(x)\phi(y)] = G(x-y)$
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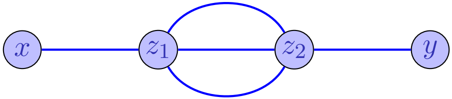

The Φ_2^4 model

- Green function: $G(x) \asymp \left|\log\|x\|\right|$
- Truncated field: $\phi_N(x) = \sum_{k \in \mathcal{K}_N} \frac{X_k}{\sqrt{\lambda_k}} e_k(x)$, $\mathcal{K}_N = \{k \in \mathbb{Z}^2 : |k| \leqslant N\}$

  Variance $C_N = \sum_{k \in \mathcal{K}_N} \frac{1}{\lambda_k} \asymp \log(N)$.
- Energy:

  $$\mathcal{H}^{\text{Wick}}_{2,\alpha,N}(\phi_N) = \int_\Lambda \Big[\|\nabla \phi_N(x)\|^2 + \frac{1}{2}\phi_N(x)^2 + \alpha \colon \phi_N(x)^4 \colon_{C_N} \Big] \mathrm{d}x$$

  where $\colon \phi_N(x)^4 \colon_{C_N} := H_4(\phi(x); C_N)$.
- Ratio of partition functions:

  $$\frac{\mathscr{Z}_{2,\alpha,N}}{\mathscr{Z}_{2,0,N}} \asymp \sum_{n \geqslant 0} \frac{(-\alpha)^n}{n!} \mathbb{E}^{\mu_{2,0,N}} \left[\left(\int_\Lambda \colon \phi_N(x)^4 \colon_{C_N} \mathrm{d}x \right)^n \right]$$

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Nelson's argument

Lemma:

Fix two cut-offs $M > N \geqslant 1$. Then for any $p > 1$ and $n \geqslant 2$, there exists a constant K_n depending only on n such that

$$\mathbb{E}^{\mu_{2,0,N}}\left[\left(\int_\Lambda :\phi_M(x)^n:_{C_M}\,\mathrm{d}x - \int_\Lambda :\phi_N(x)^n:_{C_N}\,\mathrm{d}x\right)^{2p}\right]^{\frac{1}{2p}} \leqslant K_n(2p-1)^{n/2}\frac{(\log N)^{n-2}}{N}$$

Proposition: Nelson's estimate

For any $\alpha \geqslant 0$, there exists a constant $K > 0$, indep. of N, s.t. for all $N \in \mathbb{N}$

$$0 \leqslant \mathbb{E}^{\mu_{2,0,N}}\left[\exp\left\{-\alpha \int_\Lambda :\phi_N(x)^4:\mathrm{d}x\right\}\right] \leqslant K$$

Proposition: Asymptotic series

For every $n \geqslant 0$ and $N \geqslant 1$, there exists M_n s.t.

$$\left|\frac{\mathscr{Z}_{2,\alpha,N}}{\mathscr{Z}_{2,0,N}} - \sum_{m=0}^{n}\frac{(-\alpha)^m}{m!}\mathbb{E}^{\mu_{2,0,N}}\left[\left(\int_\Lambda :\phi_N(x)^4:\mathrm{d}x\right)^m\right]\right| \leqslant M_n\alpha^{n+1}$$

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For any $\alpha \geqslant 0$, there exists a constant $K > 0$, indep. of N, s.t. for all $N \in \mathbb{N}$

$$0 \leqslant \mathbb{E}^{\mu_{2,0,N}}\Big[\exp\Big\{-\alpha \int_\Lambda :\phi_N(x)^4:\mathrm{d}x\Big\}\Big] \leqslant K$$

Proposition: Asymptotic series

For every $n \geqslant 0$ and $N \geqslant 1$, there exists M_n s.t.

$$\left|\frac{\mathscr{Z}_{2,\alpha,N}}{\mathscr{Z}_{2,0,N}} - \sum_{m=0}^{n} \frac{(-\alpha)^m}{m!}\mathbb{E}^{\mu_{2,0,N}}\Big[\Big(\int_\Lambda :\phi_N(x)^4:\mathrm{d}x\Big)^m\Big]\right| \leqslant M_n \alpha^{n+1}$$

Nelson's argument

Lemma:

Fix two cut-offs $M > N \geqslant 1$. Then for any $p > 1$ and $n \geqslant 2$, there exists a constant K_n depending only on n such that

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The Φ^4_3 model

- Green function: $G(x) \asymp \dfrac{1}{\|x\|}$, $G_N(x) = \displaystyle\sum_{k \in \mathcal{K}_N} \frac{1}{\lambda_k} e_k(x) \asymp \frac{1}{\|x\| + N^{-1}}$

Theorem: Renormalisation of the Φ^4_3 model

Define the energy by

$$\mathcal{H}^{\Phi^4}_{3,\alpha,N}(\phi_N) = \int_\Lambda \Big[\|\nabla\phi_N(x)\|^2 + \frac{1}{2}\phi_N(x)^2 + \alpha {:}\phi_N(x)^4{:}_{C_N} + \beta_N(\alpha) {:}\phi_N(x)^2{:}_{C_N} + \gamma_N(\alpha) \Big] \,\mathrm{d}x$$

where $C_N = G_N(0) \asymp N$, and the additional counterterms are

$\beta_N(\alpha) = -48\alpha^2 \Pi_N\Big($ $\Big)$

$\gamma_N(\alpha) = 12\alpha^2 \Pi_N\Big($ $\Big) - 288\alpha^3 \Pi_N\Big($ $\Big)$

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Proposition: [Weinberg]

Assume $G_N(x) \asymp (\|x\| + N^{-1})^{d-2}$. If Γ satisfies $\deg(\bar\Gamma) > 0$ for any subgraph $\bar\Gamma$ of Γ, then $\Pi_N(\Gamma)$ is bounded uniformly in N.

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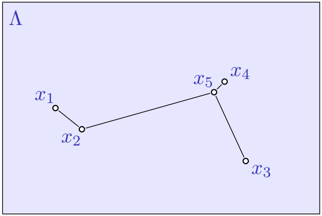

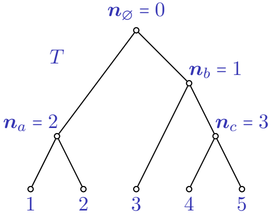

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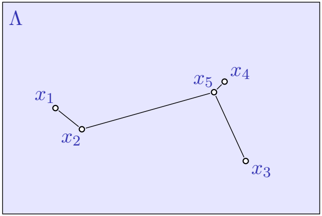

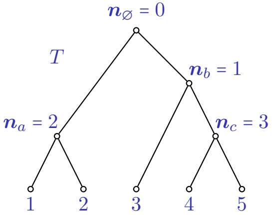

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BPHZ renormalisation

Theorem: [Bogoliubov, Parasiuk, Hepp, Zimmermann]

There exists a linear map $\mathscr{A}$, acting on Feynman diagrams, such that

$$\Pi_N(\mathscr{A}(\Gamma)) \asymp \begin{cases} N^{-\deg(\Gamma)} & \text{if } \deg(\Gamma) < 0\;, \\ \log(N)^{\zeta} & \text{if } \deg(\Gamma) = 0\;, \end{cases}$$

for a finite integer ζ, while $\Pi_N(\mathscr{A}(\Gamma))$ is bounded uniformly in N if $\deg(\Gamma) > 0$.

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$\mathscr{A}\Big($ $\Big) = -$ $+$ $\cdot$

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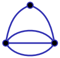
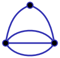
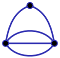
$\mathscr{A}\Big($  $\Big) = -$  $+$  $\cdot$ 

$\Pi_N\Big(\mathscr{A}$  $\Big) =$

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Wick map

- $\dfrac{\mathscr{Z}_{3,\alpha,N}}{\mathscr{Z}_{3,0,N}} = \mathbb{E}^{\mu_{3,0,N}}\left[e^{-\alpha\mathbf{X}-\beta\mathbf{Y}-\gamma}\right]$, $\mathbf{X} = \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $\mathbf{Y} = \int_\Lambda :\phi(x)^2: \mathrm{d}x$

Theorem: [B, Klose, Tapia 2025]

The following diagram is commutative:

$$\begin{tikzcd}
e^{-\alpha\mathbf{X}} \arrow[r, mapsto, "\mathscr{P}"] \arrow[d, mapsto, "\mathrm{W}"'] & \mathscr{P}(e^{-\alpha\mathbf{X}}) \arrow[d, mapsto, "(\Pi_N\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta_{\mathrm{CK}}+\Theta"] \arrow[dr, bend left, "\Pi_N^{\mathrm{BPHZ}}+\Pi_N\Theta"] & \\
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where $\mathscr{P}$ performs pairings and projects on connected graphs, $\mathrm{W}(\mathbf{X}^n) = H_n(\mathbf{X}; -\beta\mathbf{Y})$ is Wick map, and $(\Pi_N\Theta \circ \mathscr{P})(e^{-\alpha\mathbf{X}}) = \gamma$.

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The $\Phi^4_{4-\varepsilon}$ model

- $d \geqslant 4$: the Φ^4_d model is trivial [Fröhlich, Aizenmann & Duminil-Copin]
- $3 < d < 4$: use $G(x) \asymp \|x\|^{-(d-2)}$. New subdiv. for $d > d^*_{\mathsf{m}}(n) = 4 - \frac{2}{n}$.

Graphs	Degree	Critical d	Minimal n
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	$10-3d$	$\frac{10}{3} = d^*_{\mathsf{m}}(3)$	5
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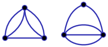
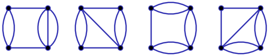
	$10 - 3d$	$\frac{10}{3} = d^*_{\mathsf{m}}(3)$	5
	$14 - 4d$	$\frac{7}{2} = d^*_{\mathsf{m}}(4)$	6

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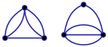
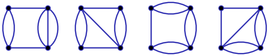
	$10 - 3d$	$\frac{10}{3} = d^*_{\mathsf{m}}(3)$	5
	$14 - 4d$	$\frac{7}{2} = d^*_{\mathsf{m}}(4)$	6

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- Cumulants: $\kappa(x^n) = \begin{cases} 0 & \text{if } n = 1 \\ y_n & \text{otherwise} \end{cases}$
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Definition: Bell polynomials

The Wick map $\mathrm{W}(t,x)$ is the generating function of Bell polynomials

$$\mathrm{W}(t,x) = \sum_{n\geqslant 0} B_n(x, -y_2, \dots, -y_n)\frac{t^n}{n!}$$

Combinatorial interpretation:

$B_5(x, y_2, y_3, y_4, y_5) = x^5 + 10x^3y_2 + 15xy_2^2 + 10x^2y_3 + 10y_2y_3 + 5xy_4 + y_5$

- 15 ways of partitioning $[\![1,5]\!]$ into 3 sets of sizes 1, 2, and 2
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