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Exercise 1: MLE for Gaussian graphical models
Let $x_1, \ldots, x_n$ be i.i.d. observations of a multivariate Gaussian $\mathbf{X}\ N_d(0, \Sigma)$. The log-likelihood in $K = \Sigma^{-1}$ is

$$\ell(K, S) \propto \log\det(K) - \operatorname{tr}(KS),$$

where $S := \frac{1}{n}\sum_{t=1}^{n} \mathbf{x}_t\mathbf{x}_t^T$ is the sample covariance. Let $G = (V, E)$ be an undirected graph. Convince yourself that the MLE in the graphical model wrt G, that is

$$\widehat{K} = \operatorname{argmax} \ell(K; S) \quad \text{s.t. } K_{ij} = 0 \quad \forall ij \notin E,$$

can be described by

$$\begin{cases} \widehat{\Sigma}_{ij} = S_{ij} & \forall ij \in E \text{ and } \forall i = j, \\ \widehat{K}_{ij} = 0 & \forall ij \notin E. \end{cases}.$$

Does this system have a unique solution?

Exercise 2: Faithfulness of linear SEMs

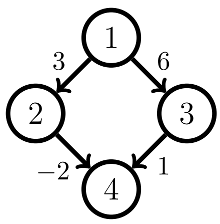


Figure 1: Diamond graph with edge weights.

Show that a linear structural equation model with the edge weights above is not faithful to its DAG.

Exercise 3: Graphical lasso example
Open the file `exercise3.R` and consider the following tasks/ questions:

- Describe the dataset. Use the documentation in the package.
- Which graphical structure do you expect?
- Try different penalty values. What do you observe?
- How would you compare the different estimators?
- Bonus: Try a validation approach. Separate the data into training and validation data and evaluate the performance of different estimates.

Exercise 4: Conditional independence

In probability and graphical models, a **collider** is a variable that is the *common effect* of two (or more) random variables.

$$X_1 \longrightarrow Z \longleftarrow X_2$$

The **collider effect** refers to the phenomenon where

$$X_1 \perp\!\!\!\perp X_2, \qquad X_1 \not\perp\!\!\!\perp X_2 \mid Z.$$

In words: conditioning on a common effect (or one of its consequences) can *create dependence* between otherwise independent variables.

Consider a factory with two machines. The factory stops if at least one of the machines fails. Define the following random variables:

1. X_1: "Machine 1 is working";

2. X_2: "Machine 2 is working";

3. Z: "The factory stops working".

Assume X_1 and X_2 are independent and each equals 1 (working) with probability 0.5. Show that $X_1 \not\perp\!\!\!\perp X_2 \mid Z$.

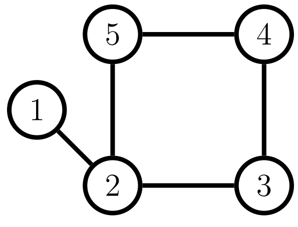
Exercise 5: Graph theory, see Hastie et al. [2009, Exercise 17.2]

Consider a random vector (X_1, X_2, X_3, X_4). In each case, draw a graph that satisfies the following conditional independence relations:

- $X_1 \perp\!\!\!\perp X_3 \mid X_2$ and $X_2 \perp\!\!\!\perp X_4 \mid X_3$;
- $X_1 \perp\!\!\!\perp X_4 \mid X_2, X_3$ and $X_2 \perp\!\!\!\perp X_4 \mid X_1, X_3$
- $X_1 \perp\!\!\!\perp X_4 \mid X_2, X_3$, $X_1 \perp\!\!\!\perp X_3 \mid X_2, X_4$ and $X_3 \perp\!\!\!\perp X_4 \mid X_1, X_2$

Exercise 6: Pairwise and global Markov property

Consider the graph below. List all the independences implied by the **pairwise** Markov property. Give one conditional independence that follows from the **Global** Markov Property.



References

Trevor Hastie, Robert Tibshirani, Jerome Friedman, et al. The elements of statistical learning, 2009.